independent unknowns of the problem as we please, we may often succeed in carrying through the treatment of a problem by transformation into free functions; for an unknown number may be chosen quite arbitrarily in all its n coefficients, and each of the following unknowns looses, as a function of the observations, only an arbitrary coefficient in comparison to the preceding one; even the n^{th} unknown can still get an arbitrary factor. Altogether are $\frac{1}{2}n(n+1)$ of the n^2 coefficients of these transformations arbitrary.

But if the problem does not admit of any solution through a transformation into free functions, the mean errors for the several unknowns, no matter how many there may be, can be computed only in such a way that each of the sought numbers are directly expressed as a linear function of the observations. The same holds good also when the laws of errors of the observations are not typical, and we are to examine how it is with λ_3 and the higher half-invariants in the laws of errors of the sought functions.

Still greater importance, nay a privileged position as the only legitimate proceeding, gets the transformation into a complete set of free functions in the over-determined problems, which are rejected as self-contradictory in exact mathematics. When we have a collection of observations whose number is greater than the number of the independent unknowns of the problem, then the question will be to determine laws of actual errors from the standpoint of the observations. We must mediate between the observations that contradict one another, in order to determine their mean numbers, and the discrepancies themselves must be employed to determine their mean deviations, etc. But as we have not to do with repetitions, the discrepancies conceal themselves behind the changes of the circumstances and require transformations for their detection. All the functions of the observations which, as the problem is over-determined, have theoretically necessary values, as, for instance, the sum of the angles of a plane triangle, must be selected for special use. Besides, those of the unknowns of the problem, to the determination of which the theory does not contribute, must come forth by the transformation by which the problem is to be solved.

As we shall see in the following chapters on Adjustment, it becomes of essential moment here that we transform into a system of free functions. The transformation begins with mutually free observations, and must not itself introduce any bond, because the transformed functions in various ways must come forth as observations which determine laws of actual errors.

X. ADJUSTMENT.

§ 48. Pursuing the plan indicated in § 5 we now proceed to treat the determination of laws of errors in some of the cases of observations made under varying or different