ON THE DEGREE OF APPROXIMATION OF CERTAIN QUADRATURE FORMULAS

By

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If $f(x)$ be a continuous function of period 2π, and if the interval under consideration, say the interval from 0 to 2π, be divided into m equal parts by the $m+1$ points $x_i = 2i\pi/m$, $i = 0, 1, 2, \ldots\ldots, m$, then the trigonometric sum of the nth order coinciding in value with $f(x)$ at the $m+1$ points x_i, or the trigonometric sum of the nth order lacking the term in $\sin nx$, is, according as $m = 2n+1$ or $m = 2n$,

$$\phi_n(x) = \tfrac{1}{2}a_0 + a_1\cos x + a_2\cos 2x + \cdots\cdots + a_n\cos nx$$
$$+ b_1\sin x + b_2\sin 2x + \cdots\cdots + b_n\sin nx$$

or

$$u_n(x) = \tfrac{1}{2}a_0 + a_1\cos x + a_2\cos 2x + \cdots\cdots + \tfrac{1}{2}a_n\cos nx$$
$$+ b_1\sin x + b_2\sin 2x + \cdots + b_{n-1}\sin(n-1)x,$$

where

$$a_k = \frac{h}{\pi}\sum_{i=1}^{m} f(x_i)\cos kx_i, \qquad h = \frac{2\pi}{m},$$

$$b_k = \frac{h}{\pi}\sum_{i=1}^{m} f(x_i)\sin kx_i.$$

If the Fourier coefficients of $f(x)$ be denoted by

$$\alpha_k = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos kx\,dx,$$

$$\beta_k = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin kx\,dx,$$