ON THE FINITE DIFFERENCES OF A POLYNOMIAL

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In this paper an apparently new and convenient method of finding the successive finite differences of a polynomial is considered. If operationally

$$\phi(u + r_1r_2) = E^{r_1r_2}\,\phi(u) = (1 + \Delta r_1)^{r_2}\,\phi(u)$$

then for any polynomial $f(x)$ of degree "n"

$$\begin{aligned}
f(x) &= p_0x^n + p_1x^{n-1} + \cdots + p_n \\
&= p_0(x + a)^n + q_{11}(x + a)^{n-1} + \cdots + q_{1n} \\
E^af(x) &= p_0(x + a)^n + p_1(x + a)^{n-1} + \cdots + p_n \\
\Delta_af(x) &= (p_1 - q_{11})(x + a)^{n-1} + (p_2 - q_{12})(x + a)^{n-2} + \cdots + (p_n - q_{1n}).
\end{aligned}$$

Similarly, if $f_1(x) = \Delta af(x)$, then

$$\begin{aligned}
f_1(x) &= (p_1 - q_{11})(x + 2a)^{n-1} + q_{22}(x + 2a)^{n-2} + \cdots + q_{2n} \\
E^af_1(x) &= (p_1 - q_{11})(x + 2a)^{n-1} + (p_2 - q_{12})(x + 2a)^{n-2} + \cdots + (p_n - q_{1n}) \\
\Delta_af_1(x) &= (p_2 - q_{12} - q_{22})(x + 2a)^{n-2} + \cdots + (p_n - q_{1n} - q_{2n})
\end{aligned}$$

and so on for the higher orders, since $\Delta_af_{s-1}(x) = \Delta_a^sf(x)$. In the practical application of this method, "a" may be conveniently taken as unity, and an abridged form of synthetic division employed. Thus, if

$$f(x) = 5x^4 + 3x^3 + 7x^2 - 2x + 3, \text{ then}$$

$$\begin{array}{rrrr|rl}
5 & +\ 3 & +\ 7 & -\ 2 & +\ 3 & = f \\
 & -\ 2 & +\ 9 & -\ \mathbf{11} & +\ 14 & \\
 & -\ 7 & +\ 16 & -\ 27 & & \\
 & -\ 12 & +\ 28 & & & \\
 & -\ 17 & & & & \\
\hline
 & 20 & -\ 21 & +\ 25 & -\ \mathbf{11} & = f_1 \\
 & & -\ 41 & +\ \mathbf{66} & -\ 77 & \\
 & & -\ 61 & +\ 127 & & \\
 & & -\ 81 & & & \\
\hline
 & & 60 & -\ 102 & +\ 66 & = f_2 \\
 & & & -\ \mathbf{162} & +\ 228 & \\
 & & & -\ 222 & & \\
\hline
 & & & \mathbf{120} & -\ \mathbf{162} & = f_3 \\
 & & & & -\ 282 & \\
\hline
 & & & & \mathbf{120} & = f_4.
\end{array}$$