ON SAMPLES FROM A MULTIVARIATE NORMAL POPULATION[1]

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1. **Introduction.** In this paper we shall discuss the distribution of certain functions calculated for samples drawn from a multivariate normal population. The method of solution is based on the theory of characteristic functions and presents further application of that theory to the distribution problem of statistics.[2]

We shall have occasion to refer to the multivariate normal population whose distribution law is given by

$$F(x) \equiv \pi^{-n/2} \,|\, B_{pq} \,|^{1/2} e^{-B(x-m,\, x-m)} \qquad (p, q = 1, 2, \cdots, n) \tag{1.1}$$

where $B(x - m, x - m)$ is the real, positive definite quadratic form of the $x_p - m_p$ with matrix $\|B_{pq}\|$. Here m_p is the mean in the population of the *pth* variate and $B_{pq} = \Delta_{pq}/2\sigma_p\sigma_q\Delta$ where σ_p is the standard deviation in the population of the *pth* variate; Δ is the determinant of population correlations $\rho_{pq} = \rho_{qp}$; Δ_{pq} is the co-factor of ρ_{pq} in Δ; and $|B_{pq}|$ is the determinant of the matrix $\|B_{pq}\|$.

Since the integral of (1.1) over the entire field of variation of the variables is unity, we have (using abbreviated notation)

$$\int e^{-B(x-m,\, x-m)}\, dx = \pi^{n/2} \,|\, B_{pq} \,|^{-1/2} \tag{1.2}$$

Equation (1.2) will be true if $\|B_{pq}\|$ is complex, provided its real part is symmetric and positive definite.[3]

The distribution of sample means of samples from the population (1.1) is independent of the distribution of the system of sample variances and covariances and is given by[4]

$$F_1(\bar{x}) \equiv \pi^{-n/2} \,|\, A_{pq} \,|^{1/2} e^{-A(\bar{x}-m,\, \bar{x}-m)} \tag{1.3}$$

where $A(\bar{x} - m, \bar{x} - m)$ is the real, positive definite quadratic form of the $\bar{x}_p - m_p$ with matrix $\|A_{pq}\|$. Here $\bar{x}_p = (1/N) \sum_{\alpha=1}^{N} x_{p\alpha}$ is the sample mean of the *pth*

[1] Presented to the American Mathematical Society, February 23, 1935.

[2] For more complete reference to the theory of characteristic functions as applied to statistics see S. Kullback, *Annals of Mathematical Statistics*, Vol. 5 (1934), pp. 263–307.

[3] J. Wishart and M. S. Bartlett, *Proc. Cambridge Phil. Soc.*, Vol. 29 (1933), pp. 260 ff.

[4] J. Wishart, *Biometrika*, Vol. 20 A (1928), pp. 32–52.

J. Wishart and M. S. Bartlett, *loc. cit.*