THE DISTRIBUTION LAWS OF THE DIFFERENCE AND QUOTIENT OF VARIABLES INDEPENDENTLY DISTRIBUTED IN PEARSON TYPE III LAWS[1]

By Solomon Kullback

Although the results herein described are not entirely new, it is felt that the method of solution is of interest as presenting further illustrations of the application of characteristic functions to the distribution problem of statistics (*1*).

1. Distribution law of the difference. Let $u = x - y$, where the distribution laws of x and y are independent and given respectively by

$$f_1(x) = \frac{e^{-x}\, x^{p-1}}{\Gamma(p)}\,; \quad f_2(y) = \frac{e^{-y}\, y^{q-1}}{\Gamma(q)} \quad 0 \leqq x \leqq \infty\,; 0 \leqq y \leqq \infty. \tag{1}$$

The characteristic function of the distribution law of u is given by (*1*),

$$\varphi(t) = \int_0^\infty \frac{e^{itx-x}\, x^{p-1}\, dx}{\Gamma(p)} \int_0^\infty \frac{e^{-ity-y}\, y^{q-1}\, dy}{\Gamma(q)} \tag{2}$$

$$= \frac{1}{(1 - it)^p\, (1 + it)^q}. \tag{3}$$

The distribution law of u is given by (*1*),

$$D(u) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-itu}\, dt}{(1 - it)^p\, (1 + it)^q}. \tag{4}$$

Let $1 - it = -\dfrac{z}{u}$,

$$D(u) = \frac{e^{-u}\, u^{p-1}}{2^q\, 2\pi i} \int_{-u-i\infty}^{-u+i\infty} \frac{e^{-z}\, dz}{(-z)^p \left(1 + \dfrac{z}{2u}\right)^q}. \tag{5}$$

Now it may be shown that (*1*)

$$\frac{1}{2\pi i} \int_{-u-i\infty}^{-u+i\infty} \frac{e^{-z}\, dz}{(-z)^p \left(1 + \dfrac{z}{2u}\right)^q} = \frac{e^u\, (2u)^{\frac{q-p}{2}}}{\Gamma(p)}\, W_{\frac{p-q}{2}, \frac{1-p-q}{2}}\, (2u) \tag{6}$$

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