CONVEXITY PROPERTIES OF GENERALIZED MEAN VALUE FUNCTIONS[1]

By Nilan Norris

Consider the following generalized mean value functions: (1) the unit weight or simple sample form, $\phi(t) = \left(\frac{x_1^t + x_2^t + \cdots + x_n^t}{n}\right)^{\frac{1}{t}}$; in which the x_i are positive real numbers not all equal each to each, and in which t may take any real value; (2) the weighted sample form, $\omega(t) = \left(\frac{c_1x_1^t + c_2x_2^t + \cdots + c_nx_n^t}{c_1 + c_2 + \cdots + c_n}\right)^{\frac{1}{t}}$, in which the c_i are positive numbers not all equal each to each, and in which the x_i and t are restricted as in $\phi(t)$; (3) the integral form, $\theta(t) = \left[\int_{x=0}^{1} x^t dx\right]^{\frac{1}{t}}$, where $\int_{x=0}^{1} x^t dx$ exists for every real value of t; and (4) the generalized integral form $\Phi(t) = \left[\int_{x=0}^{\infty} x^t d\psi(x)\right]^{\frac{1}{t}}$, where $\psi(x)$ is a non-decreasing function integrable in the Riemann-Stieltjes sense such that $\psi(\infty) - \psi(0) = 1$, and such that $\int_{x=0}^{\infty} x^t d\psi(x)$ exists for every real value of t. The facts that all of these functions are monotonic increasing and that both $\phi(t)$ and $\omega(t)$ have two horizontal asymptotes have been previously demonstrated.[2] Although the existence of $\phi(t)$ and $\omega(t)$ has been known since 1840, there appears to have been no attempt made to investigate the behavior of the second derivatives of them.[3]

When the x_i are price relatives, production relatives, or similar data, $\phi(t)$ and $\omega(t)$ yield common types of index numbers by direct substitution of integral values of t. For any values of t such that $0 < t_1 < t_2 < \infty$, the type bias of $\phi(t_2)$ will be greater than the type bias of $\phi(t_1)$. Similarly, for any values of t such that $-\infty < t_1 < t_2 < 0$, the type bias of $\phi(t_1)$ will be greater than the type bias of $\phi(t_2)$. The second derivatives of $\phi(t)$ and $\omega(t)$ indicate whether

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[2] G. H. Hardy, J. E. Littlewood, and G. Pólya, *Inequalities* (Cambridge University Press, London, 1934), pp. 12–15; and Nilan Norris, "Inequalities among Averages," *Annals of Mathematical Statistics*, Vol. VI, No. 1, March, 1935, pp. 27–29.

[3] Jules Bienaymé, *Société Philomatique de Paris*, Extraits des procès-verbaux des séances pendant l'année 1840 (Imprimerie D'A. René et Cie., Paris, 1841), Séance du 13 juin 1840 p. 68.