# THE LARGE-SAMPLE DISTRIBUTION OF THE LIKELIHOOD RATIO FOR TESTING COMPOSITE HYPOTHESES[1]

By S. S. Wilks

By applying the principle of maximum likelihood, J. Neyman and E. S. Pearson[2] have suggested a method for obtaining functions of observations for testing what are called *composite statistical hypotheses*, or simply *composite hypotheses*. The procedure is essentially as follows: A population $K$ is assumed in which a variate $x$ ($x$ may be a vector with each component representing a variate) has a distribution function $f(x, \theta_1, \theta_2, \cdots \theta_h)$, which depends on the parameters $\theta_1, \theta_2 \cdots \theta_h$. A *simple hypothesis* is one in which the $\theta$'s have specified values. A set $\Omega$ of admissible hypotheses is considered which consists of a set of simple hypotheses. Geometrically, $\Omega$ may be represented as a region in the $h$-dimensional space of the $\theta$'s. A set $\omega$ of simple hypotheses is specified by taking all simple hypotheses of the set $\Omega$ for which $\theta_i = \theta_{0i}$, $i = m + 1, m + 2, \cdots h$.

A random sample $O_n$ of $n$ individuals is considered from $K$. $O_n$ may be geometrically represented as a point in an $n$-dimensional space of the $x$'s. The probability density function associated with $O_n$ is

$$P = \prod_{\alpha=1}^{n} f(x_\alpha, \theta_1, \theta_2, \cdots \theta_h) \tag{1}$$

Let $P_\Omega(O_n)$ be the least upper bound of $P$ for the simple hypotheses in $\Omega$, and $P_\omega(O_n)$ the least upper bound of $P$ for those in $\omega$. Then

$$\lambda = \frac{P_\omega(O_n)}{P_\Omega(O_n)} \tag{2}$$

is defined as the likelihood ratio for testing the composite hypothesis $H$ that $O_n$ is from a population with a distribution characterized by values of the $\theta_i$ for some simple hypothesis in the set $\omega$. When we say that $H$ is true, we shall mean that $O_n$ is from some population of the set just described. In most of the cases of any practical importance, $P$ and its first and second derivatives with respect to the $\theta_i$ are continuous functions of the $\theta_i$ almost everywhere in a certain region of the $\theta$-space for almost all possible samples $O_n$. We shall only consider the case in which $P_\Omega(O_n)$ and $P_\omega(O_n)$ can be determined from the first and second order derivatives with respect to the $\theta$'s.

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[1] Presented to the American Mathmatical Society, March 26, 1937.

[2] Phil. Trans. Roy. Soc. London, Ser. A, Vol. 231, p. 295.