THE TRANSFORMATION OF STATISTICS TO SIMPLIFY THEIR DISTRIBUTION*

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1. **Introduction.** The custom of regarding a result as significant if it exceeds two or three times its standard error has now given way among informed statisticians to a consideration of the exact probabilities associated with the distribution of the statistic in question. For example, in such problems as that of examining the significance of the difference between the means of two samples, particularly small samples, it is no longer adequate to regard the difference of means, divided by the sample estimate of its standard error, as normally distributed. The significance of this ratio, "Student's ratio," is judged instead by the value of

$$P = 2 \int_t^\infty \phi_n(z)\, dz \tag{1}$$

where n is the number of degrees of freedom entering into the estimate of variance, and

$$\phi_n(z) = \frac{1}{\sqrt{\pi n}} \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)} \frac{1}{\left(1 + \frac{z^2}{n}\right)^{\frac{1}{2}(n+1)}}. \tag{2}$$

If the probability law underlying the observations themselves is normal, and they are independent, P is the exact probability of the value of t obtained being equalled or exceeded on the hypothesis that there is no real difference between the means.

Methods of approximating P have been studied by R. A. Fisher[1] and by W. A. Hendricks,[2] and tables have been presented by Student[3] and Fisher.[4] Nevertheless, the practical statistician will very frequently wish to make judgments of significance without stopping to consult a table, or laboriously to compute P, and will tend to revert to the former inaccurate but convenient practice of treating t as normally distributed with unit variance. The essential

* Presented at the joint meeting at Indianapolis of the American Mathematical Society and the Institute of Mathematical Statistics, December 30th, 1937.

[1] *Expansion of Student's Integral in Powers of* n^{-1}. Metron, vol. 5 (1925).

[2] Annals of Mathematical Statistics, vol. 7 (1936), pp. 210-221.

[3] *New Tables for Testing the Significance of Observations.* Metron, vol. 5 (1925).

[4] *Statistical Methods for Research Workers,* Oliver and Boyd, 1925–1936. Tables IV and VI.