NOTES ON THE DISTRIBUTION OF THE GEOMETRIC MEAN[1]

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There are two transformation theorems which apply particularly well to the distribution of a product and therefore to the distribution of the geometric mean of a sample. Both are implicit in the known theory of the transformation of integrals, but it is useful to state them in forms which are especially adapted to probability theory. Several examples will be considered in which distributions of the geometric mean will be derived by using these theorems.

The first theorem may be stated as

Theorem A: *Let the point set q in an N-dimensional u-space be defined so that in q a given function of the u's, $F(u_1, u_2 \cdots u_N)$ has the property that*

$$\xi \leq F < \xi + d\xi. \tag{1}$$

Let $\bar{q}$ be the elementary volume of the point set q defined as an N-tuple integral

$$\int_q du_1 \cdots du_N$$

taken over q, having a value of order $d\xi$. Let

$$u_i = \theta(t_i), \qquad i = 1, 2, \cdots N \tag{2}$$

be continuous and differentiable monotonic functions of the t's with unique inverses

$$t_i = \theta^{-1}(u_i). \tag{3}$$

Let r be the point set in t-space corresponding to q in u-space under the transformation (2) with elementary volume given by the integral

$$\bar{r} = \int_r dt_1 \cdots dt_N. \tag{4}$$

If $J(\xi)$ is defined as $\frac{dt_1}{du_1} \cdots \frac{dt_N}{du_N}$ at a point in q for which $F = \xi$, and if, for all points in q,

$$\left| \frac{dt_1}{du_1} \cdots \frac{dt_N}{du_N} - J(\xi) \right| < M \cdot d\xi. \tag{5}$$

When M is a constant, independent of q, then the volume $\bar{r}$, is, except for terms of order $(d\xi)^2$, given by

$$\bar{q}|J(\xi)|. \tag{6}$$

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