ON THE PROBABILITY THEORY OF ARBITRARILY LINKED EVENTS

BY HILDA GEIRINGER

1. **Introduction**. The classical Poisson problem can be stated as follows: Let $p_1, p_2, \cdots p_n$ be the probabilities of n *independent* events $E_1, E_2, \cdots E_n$ respectively; i.e. the probability of the simultaneous occurrence of E_i and E_j is equal to $p_i p_j$, that of E_i, E_j, E_k is equal to $p_i p_j p_k$ and so on. We seek the probability $P_n(x)$ that x of the events shall occur. If, $p_1 = p_2 = \cdots = p_n$ the problem is known as the Bernoulli problem.

More generally the n events may be regarded as *dependent*. Let p_{ij} be the probability of the simultaneous occurrence of E_i and E_j; p_{ijk} that of E_i, E_j, E_k and finally $p_{12\cdots n}$ that of $E_1, E_2, \cdots E_n$. There shall arise again the problem of determining the probability $P_n(x)$ that x of the n events will take place.[1] Furthermore the asymptotic behaviour of $P_n(x)$ for large n can be studied; and we shall especially be interested in the problem of the convergence of $P_n(x)$ towards a normal distribution or a Poisson distribution.

Even in the general case which we just explained, the sums

$$S_1 = \sum_{i=1}^{n} p_i, \qquad S_2 = \sum_{i,j=1}^{n} p_{ij}, \cdots S_n = p_{12\cdots n}$$

of our probabilities differ only by constant factors from the *factorial moments* $M_n^{(1)}, M_n^{(2)}, \cdots M_n^{(n)}$ of $P_n(x)$. For we have

$$S_\nu = \frac{1}{\nu!} M_n^{(\nu)} = \frac{1}{\nu!} \sum_{x=\nu}^{n} x(x-1) \cdots (x-\nu+1) P_n(x).$$

Starting from this remark the author has, in earlier papers, [8, 9, 10] established a theory of the asymptotic behaviour of $P_n(x)$, making use of the theory of moments. The criterion for the convergence of $P_n(x)$ towards the normal—or the Poisson—distribution consists of certain conditions[2] which the S_ν must satisfy.

In the following section a concise statement of the whole problem will be given, independently of the author's earlier publications. For the convergence towards the normal distribution we shall be able to establish a theorem under wider conditions in a manner which seems to be simpler. Finally, some applications of the theory will be considered.

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[1] See, for instance, references [1]–[7] at end of paper.

[2] Using the "theorem of the continuity of moments," Professor v. Mises [11] established sufficient conditions for the convergence of $P_n(x)$ towards a Poisson distribution in the case of the problem of "iterations." However, his reasoning can be applied to the general case without much difficulty.