NOTES ON HOTELLING'S GENERALIZED T

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1. Frequency Distribution When the Hypothesis Tested is Not True

a. The Problem. Let the simultaneous elementary probability law of the $k(f + 1)$ variables z_i and z'_{ir} ($i = 1, 2, \cdots, k; r = 1, 2, \cdots, f$) be

$$(1) \quad p(z, z') = (\sqrt{2\pi})^{-k(f+1)} \mid C \mid^{\frac{1}{2}(f+1)} \exp \left[-\frac{1}{2} \sum_{i,j=1}^{k} c_{ij} \{ (z_i - \zeta_i)(z_j - \zeta_j) + v'_{ij} \} \right],$$

where

$$v'_{ij} = \sum_{r=1}^{f} z'_{ir} z'_{jr} \qquad (i, j = 1, 2, \cdots, k)$$

C stands for the matrix $\| c_{ij} \|$ and $\mid C \mid$, the corresponding determinant. It is required to find the elementary probability law of the statistic

$$T = \mid V' \mid^{-1} \sum_{i,j=1}^{k} V'_{ij} z_i z_j,$$

where $\mid V' \mid = \mid v'_{ij} \mid$ and V'_{ij} denotes the cofactor of the element v'_{ij} in the matrix $\| v'_{ij} \|$.

The quantity fT is a generalization of "Student's" t considered by Hotelling [1]*. It is an appropriate criterion to test the hypothesis, say H_0, that the ζ_i in the parent population as given by (1) all vanish. The distribution of T when the hypothesis H_0 is true has already been obtained by Hotelling. But our knowledge of the test is hardly complete unless we know also the distribution of T when the ζ_i do not all vanish. Indeed, only such a knowledge can enable us to control the risk of error of the second kind, i.e. of failure to detect that H_0 is untrue [3, 4].

b. The Solution. Let H be a $k \times k$ non-singular matrix such that $H'CH = I$, the unit matrix, where H' denotes the transposed matrix of H. Let the sets of variables $(z_1, z_2, \cdots, z_k)$ and $(z'_{1r}, z'_{2r}, \cdots, z'_{kr})(r = 1, 2, \cdots f)$ be subject to the same collineation by means of H, so that

$$\| z_1, z_2, \cdots, z_k \| = \| t_1, t_2, \cdots, t_k \| \cdot H'$$

$$\| z'_{1r}, z'_{2r}, \cdots, z'_{kr} \| = \| t'_{1r}, t'_{2r}, \cdots, t'_{kr} \| \cdot H' \quad (r = 1, 2, \cdots, f)$$

where the t_i and t'_{ir} are the new variables. Let further the quantities τ_i be defined by

* References are given at the end of the paper.