ON AN INTEGRAL EQUATION IN POPULATION ANALYSIS

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I

A fundamental equation in population analysis rests on the following considerations: Of the persons born a years ago a certain fraction $p(a)$, ascertainable for example by means of a life table, survives to age a, and forms the a-year-old contingent of the existing population. A similar remark applies to every age of life. If, therefore, we denote by $N(t)$ the number of the population at time t, and by $B(t)$ the annual rate of births at the same time, and if we are dealing with a *closed* population, that is, one exempt from immigration and emigration, then, evidently,

$$N(t) = \int_0^\infty B(t-a)p(a)\,da. \tag{1}$$

In general $p(a)$ may be a function of t also, but we shall here consider primarily the case where $p(a)$ does not contain t explicitly.

The function $p(a)$ being known (from a life table), if $B(t)$ is given as a function of t, then $N(t)$ follows by direct integration of the right hand member of (1).

If, on the contrary, $N(t)$ is given, and $B(t)$ is to be determined, a special problem arises. On a former occasion[1] I have given a solution for cases in which the function $N(t)$ is given or can be expanded in the form of a series proceeding in ascending powers of e^{rt}, where r is constant; and, more particularly, for the case in which $N(t)$ is the logistic function

$$N(t) = \frac{N_\infty}{1+e^{-rt}} = N_\infty(e^{rt} - e^{2rt} + e^{3rt} - \cdots). \tag{2}$$

Although $N(t)$ is expanded in an exponential series in the process of obtaining the solution by this method, in the final result these terms are reunited, and only the original function $N(t)$ as such, together with its derivatives, appears. This suggests that it should be possible to obtain the result by a more direct route, retaining the function in its original form throughout the process. This is indeed the case, as will now be shown, by a method which at the same time frees us from the assumption that $N(t)$ can be represented by an exponential series in powers of e^{rt}.

This is accomplished as follows:

[1] A. J. Lotka, *Proc. Natl. Acad. Sci.*, 1929, vol. 15, p. 793; *Human Biology*, 1931, vol. 3, p. 459.

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