CONFIDENCE LIMITS FOR CONTINUOUS DISTRIBUTION FUNCTIONS[1]

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1. **Introduction.** The theory of confidence limits for unknown parameters of distribution functions has been considerably developed in recent years. This theory assumes that there is given a family F of systems of n stochastic variables $X_1(\theta_1, \cdots, \theta_k), \cdots, X_n(\theta_1, \cdots, \theta_k)$ depending upon k parameters $\theta_1, \cdots, \theta_k$ and such that the distribution function of every element of F is known.

For the case $k = 1$, for example, this theory proceeds as follows:

Denote by E an n-tuple $x_1, \cdots, x_n$ of observed values of the stochastic variables $X_1(\theta), \cdots, X_n(\theta)$ of which we know only that they constitute a system which is an element of F. E can be represented as the point $x_1, \cdots, x_n$ in an n-dimensional Euclidean space. Let there be given a positive number α, $0 < \alpha < 1$. Then to each pair E, α there is constructed a θ-interval, $[\underline{\theta}(E, \alpha), \bar{\theta}(E, \alpha)]$ with the following property: If we were to draw a sample from the system $X_1(\theta), \cdots, X_n(\theta)$, the probability is exactly α that we shall get a system of observations $E = x_1, \cdots, x_n$ such that the interval corresponding to E, α will include θ (i.e., that $\underline{\theta}(E, \alpha) \leq \theta \leq \bar{\theta}(E, \alpha)$).

In this paper we do not limit ourselves to a family of systems of n stochastic variables depending upon a finite number of parameters, but consider the family G of all systems of n stochastic variables $X_1, \cdots, X_n$ subject only to the condition that $X_1, \cdots, X_n$ are independently distributed with the same continuous distribution function.

Let E be the point in an n-dimensional Euclidean space which corresponds to the observed values $x_1, \cdots, x_n$ of the n stochastic variables $X_1, \cdots, X_n$ of which we know only that they constitute an element of the family G, i.e., that they are independently distributed with the same continuous distribution function. Let us denote their distribution function by $f(x)$; the probability that $X_i < x$ is $f(x)$, $i = 1, \cdots, n$. Let α be a number such that $0 < \alpha < 1$. To each pair E, α we shall construct two functions, $\bar{l}_{E,\alpha}(x)$ and $\underline{l}_{E,\alpha}(x)$, with the following property: The probability is α that, if we were to draw a sample from the system $X_1, \cdots, X_n$, we would get a system of observations $E = x_1, \cdots, x_n$ such that $f(x)$ lies entirely between $\bar{l}_{E,\alpha}(x)$ and $\underline{l}_{E,\alpha}(x)$ (i.e., that $\underline{l}_{E,\alpha}(x) \leq f(x) \leq \bar{l}_{E,\alpha}(x)$ for all x). We shall call $\bar{l}_{E,\alpha}(x)$ and $\underline{l}_{E,\alpha}(x)$ the upper and lower confidence limits, respectively, corresponding to the confidence coefficient α.

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