

THE LENGTH OF THE CYCLES WHICH RESULT FROM THE GRADUATION OF CHANCE ELEMENTS

BY EDWARD L. DODD

1. **Introduction.** Eugen Slutsky¹ found that under certain conditions repeated summations of chance elements lead to a sinusoidal configuration. Generalizations were made by V. Romanovsky.² A more recent paper by Slutsky³ has appeared, summarizing his original Russian memoir, and making extensions. Contributions to this subject have also been made by H. E. Jones,⁴ E. J. Moulton,⁵ and A. Wald.⁶

Readers who wish to get into touch with recent literature on periodicity are referred to two excellent books, that of Karl Stumpff⁷ with bibliography of 319 references, and that of Herman Wold,⁸ with bibliography of nearly 70 references.

In this paper, I deal with the wavy configuration resulting from a *single* application of a specified graduation formula. For this purpose, only linear operators are considered. For actual graduation it is customary to require that the sum of the coefficients or "weights" be equal to unity. But for the present purpose, this requirement is irrelevant. For example, summing and averaging are here essentially identical. The graduation formula considered may or may not be the combination of simple summations or averages. Indeed, formulas preferred by actuaries and statisticians include terms with *negative* coefficients; and thus involve an operation other than addition. F. R. Mac-

¹ Eugen Slutsky, "Sur un théorème limite relatif aux series des quantités éventuelles." *Comptes Rendus*, Vol. 185 (1927) pp. 169-171.

² V. Romanovsky, "Généralisations d'un théorème de M. E. Slutsky." *Comptes Rendus*, Vol. 192(1931) pp. 718-721. "Sur la loi sinusoidale limite." *Rendiconto Circolo Mathematico di Palermo*, Vol. 56 (1932) pp. 82-111. "Sur une généralisation de la loi sinusoidale limite." *Ibid.*, Vol. 57 (1933) pp. 130-136.

³ E. Slutsky, "The summation of random causes as a source of cyclic processes." *Econometrica*, Vol. 5 (1937) pp. 105-146.

⁴ H. E. Jones, "The theory of runs applied to time series," *Report of Third Annual Conference of Cowles Commission for Research in Economics* (1937) pp. 33-36. This abstract itself does not include reference to repetitions, mentioned by Moulton and Wald.

⁵ E. J. Moulton, "The periodic function obtained by repeated accumulation of a statistical series. *American Mathematical Monthly*, Vol. 45 (1938), pp. 583-586.

⁶ A. Wald, "Long cycles as a result of repeated integration." *American Mathematical Monthly*, Vol. 46 (1939), pp. 136-141.

⁷ Karl Stumpff, *Grundlagen und Methoden der Periodenforschung*, Berlin, 1937, Julius Springer.

⁸ Herman Wold, *A Study in the Analysis of Stationary Time Series*. Uppsala, 1938, Almqvist and Wiksells.