LIMITING DISTRIBUTIONS OF QUADRATIC AND BILINEAR FORMS[1, 2]

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1. Introduction. In a previous paper [15], several generalizations of the theorem of Fisher, [6, p. 97] and Cochran, [2, p. 178] on the joint distribution of quadratic forms in normally and independently distributed random variables were derived. The chief purpose of this paper is a demonstration that the Fisher-Cochran theorem and its generalizations are valid in the limit under conditions completely analogous to those under which the Laplace-Liapounoff theorem holds. Applications to the analysis of variance, periodogram analysis and multivariate analysis are discussed.

Our general procedure will be to find algebraic conditions on the matrices of quadratic and bilinear forms which enable us to assert that the limiting distributions of these forms are those which they would have had if the variables, the squares or products of which appear in their canonical forms, had been normally and independently distributed.[3] One thing which makes this possible is the fact that many frequently used quadratic and bilinear forms have the same rank no matter what may be the number of variables of which they are functions. For example, the rank of the square of the arithmetic mean, $\bar{x}_n$, where

$$\bar{x}_n = \frac{1}{n}(x_1 + \cdots + x_n),$$

is one for all values of n. In this case the quadratic form,

$$\frac{1}{n^2}\sum_{\mu,\nu=1}^{n} x_\mu x_\nu,$$

is a function of the n variables $x_1, x_2, \cdots, x_n$.

In paragraph 2 we state the vector form of the Laplace-Liapounoff theorem and several corollaries. The joint limiting distributions of quadratic and bilinear forms are derived in paragraph 3. The final paragraph is devoted to a statement of a few applications of the theorems.

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[2] The material contained in this paper was presented in part to the American Statistical Association, December 28, 1937, and in part to the Institute of Mathematical Statistics, December 27, 1938.

[3] We shall be chiefly concerned with conditions under which the limiting distributions are not themselves normal. If the limiting distributions are normal, then generally under the conditions we state, the Laplace-Liapounoff theorem will have been directly applicable.