CONDITIONS FOR UNIQUENESS IN THE PROBLEM OF MOMENTS

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It was shown by Stieltjes [1] that in some circumstances it is possible for two different frequency distributions to have the same set of moments. For instance, the integral

$$\int z^{4n+3} e^{-z} e^{iz} \, dz$$

around a contour consisting of the positive x-axis, the infinite quadrant and the positive y-axis is seen to be zero and it follows that

$$\int_0^\infty x^n e^{-x^{\frac{1}{4}}} \sin x^{\frac{1}{4}} \, dx = 0.$$

Thus the frequency distribution

$$dF = \tfrac{1}{6} e^{-x^{\frac{1}{4}}}(1 - \lambda \sin x^{\frac{1}{4}}) \, dx \qquad \begin{array}{l} 0 \leq x \leq \infty, \\ 0 \leq \lambda \leq 1 \end{array} \tag{1}$$

has moments which are independent of λ, and equation (1) may be regarded as defining a whole family of distributions each of which has the same moments. It is easy to see that moments of all orders exist, and in fact

$$\mu_r' \text{ (about the origin)} = \tfrac{1}{6}(4r + 3)!.$$

A second example of the same kind, also due to Stieltjes, is the distribution

$$dF = \frac{1}{e^{\frac{1}{4}}\sqrt{\pi}} x^{-\log x}\{1 - \lambda \sin (2\pi \log x)\} \, dx \qquad \begin{array}{l} 0 \leq x \leq \infty, \\ 0 \leq \lambda \leq 1, \end{array} \tag{2}$$

for which

$$\mu_r' = e^{\frac{1}{4}r(r+2)}.$$

The question naturally arises, what are the conditions under which a given set of moments determines a frequency distribution uniquely? The question is of great interest to mathematicians, being closely linked with problems in the theory of asymptotic series, continued fractions and quasi-analytic functions; and it also has importance for statisticians since there is sometimes occasion to be satisfied that a problem of finding a frequency distribution has been uniquely solved by the ascertainment of its moments or semi-invariants. Stieltjes himself considered a more general problem: given a set of constants c_0,