A GENERALIZATION OF THE LAW OF LARGE NUMBERS

By Hilda Geiringer

It is well known that the law of large numbers can be established for dependent as well as for independent chance variables by using Tchebycheff's inequality [1] and assuming that the variance of the sum of the variables tends towards infinity less rapidly than n^2.

In recent years v. Mises has introduced the notion of *statistical functions* [2] and has shown that, under certain assumptions the law of large numbers is still valid if, instead of the arithmetic mean of the n observations $x_1, \cdots, x_n$ a statistical function of these observations is considered. For example in the very special case, where the n collectives which have been observed are *identical* k-valued *arithmetic* distributions with probabilities $p_1, \cdots, p_k$ corresponding to the attributes $c_1, \cdots, c_k$ and with observed relative frequencies $n_1/n, \cdots, n_k/n$ one obtains the result: It is to be expected for every $\epsilon > 0$ with a probability P_n converging towards one as $n \to \infty$, that $|f(n_1/n, \cdots, n_k/n) - f(p_1, \cdots, p_k)| < \epsilon$ under very general conditions concerning the function f.

In the present paper we shall generalize these new results so that they will apply also to collectives which are not independent.

1. Lemma concerning alternatives. Let us consider the *n-dimensional collective* consisting of a *sequence of n trials* and let us assume that the n trials are alternatives, i.e. for each trial there are only two possible results which we denote by "success," "failure," by "occurrence," "non-occurrence" or by "1," "0." The total result of the n trials is expressed by n numbers each equal to 0 or 1. Let $v(x_1, x_2, \cdots, x_n)$ be the probability of obtaining the result x_1 at the first trial, x_2 at the second one, $\cdots, x_n$ at the last one ($x_\nu = 0, 1$; $\nu = 1, \cdots, n$). In the same way we introduce $v_{12}(x, y) = \sum_{x_3, \cdots, x_n} v(x, y, x_3, \cdots, x_n)$ and generally $v_{\mu\nu}(x, y)$ as the probability that the μth result equals x, the νth equals y, ($\mu \neq \nu$), and finally let $v_\mu(x) = \sum_y v_{\mu\nu}(x, y)$ be the probability that the μth result equals x. In particular let us write

$$v_\mu(1) = p_\mu, \quad v_{\mu\nu}(1, 1) = p_{\mu\nu}, \quad (\mu, \nu = 1, \cdots, n; \mu \neq \nu)$$

p_μ being the probability of success in the μth trial and $p_{\mu\nu}$ the probability of simultaneous success both in the μth and νth trials.

The *variance* s_n^2 of the sum $(x_1 + \cdots + x_n)$ is easily found: