ON A CERTAIN CLASS OF ORTHOGONAL POLYNOMIALS

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Introduction. E. H. Hildebrandt has demonstrated the following theorem[1]: *If y is a non-identically zero solution of the Pearsonian Differential Equation,*

$$\frac{1}{y}\frac{dy}{dx} = \frac{a_0 + a_1 x}{b_0 + b_1 x + b_2 x^2} \equiv \frac{N}{D}, \qquad a_i, b_i \text{ real, then} \tag{1}$$

$$\frac{D^{n-k}}{y}\frac{d^n}{dx^n}(D^k y) \equiv P_n(k, x), \qquad n, k \text{ integers}, n \geq 0, \text{ is a} \tag{2}$$

polynomial in x of degree n at most. Hildebrandt has obtained various relations connecting the $P_n(k, x)$ and their derivatives as well as a recurrence relation.

If in (2) we set $k = n$ there results from a proper choice of N and D in (1), the classical Hermite, Laguerre, Jacobi and Legendre Polynomials. Many properties of these classical polynomials have been obtained by numerous investigators.[2]

One of the most important of these properties is that of orthogonality which can be stated as follows: *Consider a sequence of the classical polynomials* $\Phi_i(x) = x^i - S_i x^{i-1} + \cdots$. *There exists an interval* (a, b) *finite or infinite and a unique weight function* $\psi(x)$, *monotonic non-decreasing over* (a, b) *such that,*

$$\int_a^b \Phi_m(x)\Phi_n(x)\, d\psi(x) = 0, \qquad \text{for } n \neq m. \tag{3}$$

In the future we will refer to the type of orthogonality given by (3) *with* $\psi(x)$ *monotonic non-decreasing as orthogonality in the restricted sense.* In order to determine whether a given system of polynomials is orthogonal in the restricted sense we have the following theorem:[3]

Theorem 1. *In order that the sequence of polynomials* $\Phi_i(x) = x^i - S_i x^{i-1} +$

[1] E. H. Hildebrandt, "Systems of polynomials connected with the Charlier expansions, etc.," *Annals of Math. Stat.*, Vol. 2(1931), pp. 379–439.

[2] For an account of these properties as well as an extensive bibliography the reader can refer to one of two treatises viz.: J. Shohat, *Théorie Générale des Polynomes Orthogonaux de Tchebichef*, Memoriale des Sciences Mathématiques, Fascicule 66, Paris, Gauthier Villars, 1936.

Gabor Szegö, *Orthogonal Polynomials*, Am. Math. Soc., Colloquium Publications, Vol. 23, 1939.

[3] J. Shohat, "The relation of the classical orthogonal polynomials to the polynomials of Appell," *Am. Jour. of Math.*, Vol. 58(1936), pp. 454–455.