ASYMPTOTICALLY MOST POWERFUL TESTS OF STATISTICAL
HYPOTHESES

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1. Introduction. Let \( f(x, \theta) \) be the probability density function of a variate
\( x \) involving an unknown parameter \( \theta \). For testing the hypothesis \( \theta = \theta_0 \) by
means of \( n \) independent observations \( x_1, \ldots, x_n \) on \( x \) we have to choose a region
of rejection \( W_n \) in the \( n \)-dimensional sample space. Denote by \( P(W_n \mid \theta) \) the
probability that the sample point \( E = (x_1, \ldots, x_n) \) will fall in \( W_n \) under the
assumption that \( \theta \) is the true value of the parameter. For any region \( U_n \) of
the \( n \)-dimensional sample space denote by \( g(U_n) \) the greatest lower bound of
\( P(U_n \mid \theta) \). For any pair of regions \( U_n \) and \( T_n \) denote by \( L(U_n, T_n) \) the least
upper bound of

\[
P(U_n \mid \theta) - P(T_n \mid \theta).
\]

In all that follows we shall denote a region of the \( n \)-dimensional sample space
by a capital letter with the subscript \( n \).

Definition 1. A sequence \( \{W_n\}, (n = 1, 2, \ldots, \text{ad inf.}) \), of regions is said to
be an asymptotically most powerful test of the hypothesis \( \theta = \theta_0 \) on the level
of significance \( \alpha \) if \( P(W_n \mid \theta_0) = \alpha \) and if for any sequence \( \{Z_n\} \) of regions for
which \( P(Z_n \mid \theta_0) = \alpha \), the inequality

\[
\limsup_{n \to \infty} L(Z_n, W_n) \leq 0
\]

holds.

Definition 2. A sequence \( \{W_n\}, (n = 1, 2, \ldots, \text{ad inf.}) \), of regions is said to
be an asymptotically most powerful unbiased test of the hypothesis \( \theta = \theta_0 \) on the level of significance \( \alpha \) if \( P(W_n \mid \theta_0) = \lim g(W_n) = \alpha \), and if for any se-
quency \( \{Z_n\} \) of regions for which \( P(Z_n \mid \theta_0) = \lim g(Z_n) = \alpha \), the inequality

\[
\limsup_{n \to \infty} L(Z_n, W_n) \leq 0
\]

holds.

Let \( \hat{\theta}_n(x_1, \ldots, x_n) \) be the maximum likelihood estimate of \( \theta \) in the \( n \)-dimen-
sional sample space. That is to say, \( \hat{\theta}_n(x_1, \ldots, x_n) \) denotes the value of \( \theta \)

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