# DISCUSSION OF PAPERS ON PROBABILITY THEORY

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1. **Comments by R. von Mises.** Professor Doob outlines a new theory of probability starting with the following three basic conceptions. First, he uses the notion of an infinite sequence of trials or better: of an infinite sequence of numbers $x_1, x_2, x_3, \cdots$ which can be considered as the outcomes of infinitely repeated uniform experiments. Second, he introduces (in his Theorem A) the limit of the relative frequency of a particular outcome $\alpha$. Third, (in his Theorem B) the notion of place selection defined by a sequence of functions $f_n(x_1, x_2, \cdots x_{n-1})$ is employed. All these three concepts are completely strange to the so called classical theory as developed by Bernoulli, Laplace, Poisson, etc. They have been introduced and made the corner stone of probability theory in my papers published since 1919. I daresay that in no probability investigation before 1919 any of those notions even were mentioned.

This concerns what Professor Doob calls the Problem I or the purely mathematical aspect of the question. As to his Problem II or the relationship between the formal calculus and real facts Professor Doob stresses that the actual values for probabilities that enter as data into a particular argument have to be drawn from long, finite sequences of experiments. This is in complete accordance with the standpoint of my theory and in strict contradiction to the classical conception which knows only "a priori" probabilities determined by "equally likely cases."

In both theories, Professor Doob's and mine (not in the classical) a mathethematical model or picture is associated with a long sequence of uniform experiments. These models are different in both theories. My model (the "kollektiv") consists of one infinite sequence $\omega$: $x_1, x_2, x_3, \cdots$ in which the limit of the relative frequency of each possible outcome $\alpha$ exists and is indifferent to a place selection; the value of this limit is called the probability of $\alpha$.

On the other hand Professor Doob's model implies all logically possible sequences which form a space $\Omega$ and he shows that in this space a measure function can be introduced which fulfills the following conditions: (1) If $m$ is a positive integer, the set of all sequences the $m$th element of which is $\alpha$ has a measure $p_\alpha$ independent of $m$; (2) the set of all sequences in which the relative frequency of $\alpha$-results has either no limit or a limit different from $p_\alpha$ is zero; (3) if $S$ is any place selection, the set of all sequences $\omega$ for which the relative frequency of $\alpha$ in $S(\omega)$ has either no limit or a limit different from $p_\alpha$ is likewise zero; this value $p_\alpha$ is called the probability of the outcome $\alpha$. It then can be shown that a probability in this sense can be ascribed to certain events, i.e. to certain types of experiments which in some way are connected with the sequence of basic