

PROBABILITY AS MEASURE

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The following pages outline a treatment of probability suitable for statisticians and for mathematicians working in that field. No attempt will be made to develop a theory of probability which does not use numbers for probabilities. The theory will be developed in such a way that the classical proofs of probability theorems will need no change, although the reasoning used may have a sounder mathematical basis. It will be seen that this mathematical basis is highly technical, but that, as applied to simple problems, it becomes the set-up used by every statistician. The formal and empirical aspects of probability will be kept carefully separate. In this way, we hope to avoid the airy flights of fancy which distinguish many probability discussions and which are irrelevant to the problems actually encountered by either mathematician or statistician.

We shall identify as Problem I the problem of setting up a formal calculus to deal with (probability) numbers. Within this discipline, once set up, the only problems will be mathematical. The concepts involved will be ordinary mathematical ones, constantly used in other fields. The words "probability," "independent," etc. will be given mathematical meanings, where they are used.

We shall identify as Problem II the problem of finding a translation of the results of the formal calculus which makes them relevant to empirical practice. Using this translation, experiments may suggest new mathematical theorems. If so, the theorems must be stated in mathematical language, and their validity will be independent of the experiments which suggested them. (Of course, if a theorem, after translation into practical language, contradicts experience, the contradiction will mean that the probability calculus, or the translation, is inappropriate.)

The classical probability investigators did not separate Problems I and II carefully, thinking of probability numbers as numbers corresponding to events or to hypothetical truths, and always referring the numbers back to their physical counterparts. The measure approach to the probability calculus has put this approach into abstract form, and separated out the empirical elements, thus removing all aspects of Problem II. We shall explain this approach first in a simplified set-up, that which will be made to correspond (Problem II) to a repeated experiment in which the results of the n th trial can be any integer x_n between 1 and N (inclusive), in which the experiments are independent of each other, and performed under the same conditions. (The set-up will be applicable, for example, to the repeated throwing of a die.)