CONDITIONS THAT THE ROOTS OF A POLYNOMIAL BE LESS THAN UNITY IN ABSOLUTE VALUE

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1. Introduction. In econometric business cycle analysis, probability theory, and numerical mathematical computation the problem of convergence of repeated iterations arises. The solution of the difference equations defining such a process can in a wide variety of cases be shown to be stable in the sense of converging to a limit if a certain associated polynomial

$$f(x) = p_0x^n + p_1x^{n-1} + \cdots + p_n = 0, \tag{1}$$

has roots whose moduli are all less than unity.

Thus, for "timeless" linear difference equation systems of the most general type, convertible into normal form,

$$Q_i(t+1) = \sum_{j=1}^{n} a_{ij}Q_j(t), \qquad (i = 1, \cdots, n), \tag{2}$$

the polynomial is the characteristic or determinantal equation,

$$f(x) = | a_{ij} - x\delta_{ij} | = 0, \tag{3}$$

which when expanded out is of the form (1). The roots of this equation, when multiplied by suitable polynomials in t, give the exact solution of the problem in the form

$$Q(t) = \sum_{i=1}^{m} g_i(t)x_i^t, \tag{4}$$

where m is the number of distinct roots, and the g's are polynomials of degree one less than the multiplicity of the respective root. If complex roots occur, they do so in conjugate pairs and can be combined to form damped, undamped, or anti-damped harmonic terms. All terms go to zero as t approaches infinity if, and only if, the absolute value of each x is less than unity.

For non-linear systems the exact solution does not take this form, but in the neighborhood of an equilibrium point the roots of an associated polynomial, except in singular cases, do determine the stability of the system.

As far as the writer is aware, there does not appear in the literature an account of necessary and sufficient conditions for the roots of a polynomial to be less than unity in absolute value. This is in contrast to a related problem which arises in connection with the investigation of stability of dynamical systems defined by differential equations. These have associated with them a polynomial whose roots provide solutions in the form

$$g_i(t)e^{x_it}, \tag{5}$$