ON THE INTEGRAL EQUATION OF RENEWAL THEORY

By Willy Feller

Brown University

1. **Introduction.** In this paper we consider the behavior of the solutions of the integral equation

$$u(t) = g(t) + \int_0^t u(t - x)f(x)\,dx, \tag{1.1}$$

where $f(t)$ and $g(t)$ are given non-negative functions.[1] This equation appears, under different forms, in population theory, the theory of industrial replacement and in the general theory of self-renewing aggregates, and a great number of papers have been written on the subject.[2] Unfortunately most of this literature is of a heuristic nature so that the precise conditions for the validity of different methods or statements are seldom known. This literature is, moreover, abundant in controversies and different conjectures which are sometimes supported or disproved by unnecessarily complicated examples. All this renders an orientation exceedingly difficult, and it may therefore be of interest to give a rigorous presentation of the theory. It will be seen that some of the previously announced results need modifications to become correct.

The existence of a solution $u(t)$ of (1.1) could be deduced directly from a well-known result of Paley and Wiener [21] on general integral equations of form (1.1).[3] However, the case of non-negative functions $f(t)$ and $g(t)$, with which we are here concerned, is much too simple to justify the deep methods used by Paley and Wiener in the general case. Under the present conditions, the existence of a solution can be proved in a simple way using properties of completely monotone functions, and this method has also the distinct advantage of showing some properties of the solutions, which otherwise would have to be proved separately. It will be seen in section 3 that the existence proof becomes most natural if equation (1.1) is slightly generalized. Introducing the summatory functions

$$U(t) = \int_0^t u(x)\,dx, \qquad F(t) = \int_0^t f(x)\,dx, \qquad G(t) = \int_0^t g(x)\,dx, \tag{1.2}$$

[1] For the interpretation of the equation cf. section 2.

[2] Lotka's paper [8] contains a bibliography of 74 papers on our subject published before 1939. Yet it is stated that even this list "is not the result of an exhaustive search." At the end of the present paper the reader will find a list of 16 papers on (1.1) which have appeared during the two years since the publication of Lotka's paper.

[3] This has been remarked also by Hadwiger [3].