NOTES

This section is devoted to brief research and expository articles, notes on methodology and other short items.

A FURTHER REMARK CONCERNING THE DISTRIBUTION OF THE RATIO OF THE MEAN SQUARE SUCCESSIVE DIFFERENCE TO THE VARIANCE[1]

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1. **Introduction.** In our previous paper[1] it was found convenient to assume that the number m (of the variables of the quadratic form under consideration) is even. (Cf. p. 383, loc. cit.) This means that in the application to the mean square successive difference $n = m + 1$ must be odd. (Cf. p. 389, id.)

In this note we shall show that the distribution for an odd m (i.e. an even n) can be expressed by means of the distribution for an even m—the latter being already known, loc. cit.

Specifically, consider the distribution of $\gamma = \sum_{\mu=1}^{m} a_\mu x_\mu^2$, if the $x_1, \cdots, x_m$ are equidistributed over the surface $\sum_{\mu=1}^{m} x_\mu^2 = 1$. Denote the m-uplet $(a_1, \cdots, a_m)$ by A, then the distribution function of γ depends on A; denote that distribution by $\omega_A(\gamma)$. (Cf. p. 372 id., we write a_μ for the B_μ there.)

Now consider an m-uplet $A = (a_1, \cdots, a_m)$ and a p-uplet $B = (b_1, \cdots, b_p)$ and form the $m + p$-uplet $C = (a_1, \cdots, a_m, b_1, \cdots, b_p)$. Write $C = A + B$. Then we shall show that there exists a simple expression for $\omega_C(\gamma)$ in terms of $\omega_A(\gamma)$ and $\omega_B(\gamma)$.

For the specific application to the mean square successive difference, we can put $n = m + 1$, $A = (\cos (\pi\mu/n)$ for $\mu = 1, \cdots, \frac{1}{2}n - 1, \frac{1}{2}n + 1, \cdots, n - 1)$, $B = (0)$, $C = A + B = (\cos \pi\mu/n$ for $\mu = 1, \cdots, n - 1)$.

2. **The recursion formula.** We proceed as follows. $\omega_A(\gamma)$ can also be used to express the joint statistics of

$$\gamma = \sum_{\mu=1}^{m} a_\mu x_\mu^2 \quad \text{and} \quad \rho = \sum_{\mu=1}^{m} x_\mu^2,$$

or better, the volume of that part of the $x_1, \cdots, x_m$-space which corresponds to any given domain in the γ, ρ-plane. Thus the volume corresponding to a

[1] Cf. the paper by the same author, *Annals of Math. Stat.*, vol. 12(1941), pp. 367–395.

[2] Also Scientific Advisory Committee of the Ballistic Research Laboratory, Aberdeen Proving Ground.