NOTES

This section is devoted to brief research and expository articles, notes on methodology and other short items.

A FURTHER REMARK CONCERNING THE DISTRIBUTION OF THE RATIO OF THE MEAN SQUARE SUCCESSIVE DIFFERENCE TO THE VARIANCE

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1. Introduction. In our previous paper it was found convenient to assume that the number \( m \) (of the variables of the quadratic form under consideration) is even. (Cf. p. 383, loc. cit.) This means that in the application to the mean square successive difference \( n = m + 1 \) must be odd. (Cf. p. 389, id.)

In this note we shall show that the distribution for an odd \( m \) (i.e. an even \( n \)) can be expressed by means of the distribution for an even \( m \)—the latter being already known, loc. cit.

Specifically, consider the distribution of \( \gamma = \sum_{\mu=1}^{m} a_{\mu}x_{\mu}^2, \) if the \( x_1, \cdots, x_m \) are equidistributed over the surface \( \sum_{\mu=1}^{m} x_{\mu}^2 = 1. \) Denote the \( m \)-uplet \((a_1, \cdots, a_m)\) by \( A, \) then the distribution function of \( \gamma \) depends on \( A; \) denote that distribution by \( \omega_A(\gamma). \) (Cf. p. 372 id., we write \( a_\mu \) for the \( B_\mu \) there.)

Now consider an \( m \)-uplet \( A = (a_1, \cdots, a_m) \) and a \( p \)-uplet \( B = (b_1, \cdots, b_p) \) and form the \( m + p \)-uplet \( C = (a_1, \cdots, a_m, b_1, \cdots, b_p). \) Write \( C = A + B. \) Then we shall show that there exists a simple expression for \( \omega_C(\gamma) \) in terms of \( \omega_A(\gamma) \) and \( \omega_B(\gamma). \)

For the specific application to the mean square successive difference, we can put \( n = m + 1, \) \( A = (\cos \frac{\pi \mu}{n} \text{ for } \mu = 1, \cdots, \frac{3n}{2} - 1), \) \( B = (0), \) \( C = A + B = (\cos \frac{\pi \mu}{n} \text{ for } \mu = 1, \cdots, n - 1). \)

2. The recursion formula. We proceed as follows. \( \omega_A(\gamma) \) can also be used to express the joint statistics of

\[ \gamma = \sum_{\mu=1}^{m} a_{\mu}x_{\mu}^2 \quad \text{and} \quad \rho = \sum_{\mu=1}^{m} x_{\mu}^2, \]

or better, the volume of that part of the \( x_1, \cdots, x_m \)-space which corresponds to any given domain in the \( \gamma, \rho \)-plane. Thus the volume corresponding to a

\[ \text{Cf. the paper by the same author, Annals of Math. Stat., vol. 12(1941), pp. 367-395.} \]

[1] Also Scientific Advisory Committee of the Ballistic Research Laboratory, Aberdeen Proving Ground.