TABULATION OF THE PROBABILITIES FOR THE RATIO OF THE MEAN SQUARE SUCCESSIVE DIFFERENCE TO THE VARIANCE

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with a note

By John von Neumann

In recent publications von Neumann has determined the distribution of δ^2/s^2, the ratio of the mean square successive difference to the variance, for odd values of the sample size n[1] and for even values of n.[2] In this paper the probability function, i.e., the integral of the distribution, is evaluated for specific values of n.

Let x be a stochastic variable normally distributed with mean ζ and the standard deviation σ. The following customary definitions for the sample are:

the mean, $$\bar{x} = \frac{1}{n}\sum_{\mu=1}^{n} x_\mu,$$

the variance, $$s^2 = \frac{1}{n}\sum_{\mu=1}^{n} (x_\mu - \bar{x})^2,$$

and the mean square successive difference, $\delta^2 = \dfrac{1}{n-1}\sum_{\mu=1}^{n-1} (x_{\mu+1} - x_\mu)^2$. Letting $\dfrac{\delta^2}{s^2} = \dfrac{2n}{n-1}(1-\epsilon)$, von Neumann shows that the distribution of ϵ, $\omega(\epsilon)$, is symmetrical with zero mean and intercepts equal to $\pm \cos\dfrac{\pi}{n}$ (loc. cit.[1], p. 372), and that $\omega(\epsilon)$ is determined for odd values of n by

$$\frac{d^{\frac{1}{2}(n-1)-1}}{d\epsilon^{\frac{1}{2}(n-1)-1}}\,\omega(\epsilon) = \pm\frac{(\frac{1}{2}[n-1]-1)!}{\pi}\,\frac{1}{\sqrt{\prod_{\mu=1}^{n-1}\left(\epsilon - \cos\frac{\mu\pi}{n}\right)}},$$

in the odd intervals

$$\cos\frac{\pi}{n} \geqq \epsilon \geqq \cos\frac{2\pi}{n},$$

$$\frac{\cos 3\pi}{n} \geqq \epsilon \geqq \cos\frac{4\pi}{n}, \cdots, \frac{\cos(n-2)\pi}{n} \geqq \epsilon \geqq \frac{\cos(n-1)\pi}{n},$$

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[1] John von Neumann, "Distribution of the ratio of the mean square successive difference to the variance," *Annals of Math. Stat.*, Vol. 12 (1941), pp. 367–395.

[2] John von Neumann, "A further remark on the distribution of the ratio of the mean square successive difference to the variance," *Annals of Math. Stat.*, Vol. 13 (1942), pp. 86–88.