TABULATION OF THE PROBABILITIES FOR THE RATIO OF THE MEAN SQUARE SUCCESSIVE DIFFERENCE TO THE VARIANCE

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with a note

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In recent publications von Neumann has determined the distribution of \( \delta^2/\sigma^2 \), the ratio of the mean square successive difference to the variance, for odd values of the sample size \( n \) and for even values of \( n \). In this paper the probability function, i.e., the integral of the distribution, is evaluated for specific values of \( n \).

Let \( x \) be a stochastic variable normally distributed with mean \( \mu \) and standard deviation \( \sigma \). The following customary definitions for the sample are:

the mean, \( \bar{x} = \frac{1}{n} \sum_{i=1}^{n} x_i \),

the variance, \( s^2 = \frac{1}{n} \sum_{i=1}^{n} (x_i - \bar{x})^2 \),

and the mean square successive difference, \( \delta^2 = \frac{1}{n-1} \sum_{i=1}^{n} (x_{i+1} - x_i)^2 \). Letting \( \frac{\delta^2}{\sigma^2} = \frac{2n}{n-1} (1 - \epsilon) \), von Neumann shows that the distribution of \( \epsilon \), \( \omega(\epsilon) \), is symmetrical with zero mean and intercepts equal to \( \pm \cos \frac{\pi}{n} \) (loc. cit.\(^1\), p. 372), and that \( \omega(\epsilon) \) is determined for odd values of \( n \) by

\[
\frac{d^{(n-1)-1}}{d\epsilon^{(n-1)-1}} \omega(\epsilon) = \pm \frac{\left(\frac{1}{2}[n - 1] - 1 \right) !}{\pi} \frac{1}{\sqrt{\prod_{j=1}^{n-1} \left( \epsilon - \cos \frac{\mu \pi}{n} \right)}}\]

in the odd intervals

\[
\frac{\cos \frac{\pi}{n}}{n} \geq \epsilon \geq \cos \frac{2\pi}{n},
\]

\[
\frac{\cos \frac{3\pi}{n}}{n} \geq \epsilon \geq \cos \frac{4\pi}{n}, \ldots, \frac{\cos \frac{(n-2)\pi}{n}}{n} \geq \epsilon \geq \cos \frac{(n-1)\pi}{n},
\]
