LINEAR RESTRICTIONS ON CHI-SQUARE

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Chi-square is a statistic widely used in statistical analysis. It is usually of the form,

$$\chi^2 = \sum_1^n \chi_i^2 \tag{1}$$
$$= \sum_1^n \left(\frac{x_i - m_i}{\sigma_i}\right)^2,$$

where the x_i's are independent normally distributed variables drawn from populations with respective means and standard deviations, m_i and σ_i. In practical problems the independence of the x_i's is often modified by placing restrictions on the χ_i's in order to estimate the m_i's or σ_i's. It is well known that if m such restrictions which are linear and homogeneous (also algebraically independent) are placed on the χ_i's, then the resulting chi-square, (1), is distributed according to the chi-square distribution with $n - m$ degrees of freedom. The purpose of this paper is to study the case where the restrictions are not necessarily homogeneous.

1. Geometrical development. The χ_i's of equation (1) may be considered as co-ordinates in an n-dimensional space. Equation (1) represents a sphere in such a space with its center at the origin and with radius, χ. We should like to determine the distribution of χ^2. First, since the χ_i's are independent, we may form their joint distribution,[1]

$$F(\chi_1, \chi_2, \cdots \chi_n)\, dV = K\Pi_j e^{-\frac{1}{2}\chi_j^2}\, d\chi_j \tag{2}$$
$$= Ke^{-\frac{1}{2}\Sigma\chi_j^2}\, d\chi_1\, d\chi_2 \cdots d\chi_n$$
$$= Ke^{-\frac{1}{2}\chi^2}\, dV.$$

We may change the variable in (2) to χ^2 if we can determine dV. Since the n-dimensional sphere represented by equation (1) has a volume proportional to χ^n, we may write

$$dV = Kd(\chi^2)^{\frac{1}{2}n}$$
$$= K(\chi^2)^{\frac{1}{2}n-1}\, d\chi^2.$$

Substituting this value in the distribution (2) we obtain for the distribution of chi-square,

$$F(\chi^2)\, d\chi^2 = K(\chi^2)^{\frac{1}{2}n-1} e^{-\frac{1}{2}\chi^2}\, d\chi^2,$$

which is the usual form of the chi-square distribution for n degrees of freedom.

[1] The letter K will be used throughout as a constant, not necessarily the same constant from equation to equation.

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