ADDITIVE PARTITION FUNCTIONS AND A CLASS OF STATISTICAL HYPOTHESES

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1. Introduction. The purpose of the first part of this paper is to prove several theorems about a class of functions of partitions which are additive in structure and subject to mild restrictions. These theorems may be regarded as contributions to the theory of numbers, but if one makes certain assignments of probabilities to the partitions the theorems may be expressed as statements about asymptotic distributions. It is in this latter, probabilistic language, that we shall carry out the proofs, for the following reasons. The discussion will be more concise and certain circumlocutions will be avoided. The theorems have statistical application and a number of theorems discussed recently in statistical literature are corollaries of one of our theorems.

In the second part of this paper the theory of testing statistical hypotheses where the form of the distribution functions is totally unknown and only continuity is assumed, will be discussed. The exact extension of the likelihood ratio criterion to this case will be given. Approximations to the application of this criterion in two problems will be proposed, one of which applies the results mentioned above. Lastly, in connection with the second problem, a combinatorial problem will be solved which is new and has interest per se.

2. Partitions of a single integer. Let n be a positive integer and $A = (a_1, a_2, \cdots, a_s)$ be any sequence of positive integers a_i $(i = 1, 2, \cdots, s)$, where $\sum_{i=1}^{s} a_i = n$, and s may be any integer from 1 to n. Two sequences A which have different elements or the same elements arranged in different order are to be considered distinct, so it is easy to see that there are 2^{n-1} sequences A. We shall consider the sequence A as a stochastic variable and assign to all sequences A the same probability, which is therefore 2^{-n+1}. Let r_j be the number of elements a in A which equal j $(j = 1, 2, \cdots, n)$, so that r_j is a stochastic variable. Let k be an integer $\leq n$. Then the joint distribution of the stochastic variables $r_1, r_2, \cdots, r_k$ is given as follows: The probability that $r_i = b_i$ $(i = 1, 2, \cdots, k)$ is

$$2^{-n+1}\left(\sum_{r=1}^{n} \sum \frac{r!}{(b_1)!(b_2)! \cdots (b_k)!(r_{(k+1)})! \cdots (r_n)!}\right), \tag{2.1}$$

where the inner summation is carried out over all sets of non-negative integers $r_{(k+1)}, \cdots, r_n$ such that

$$b_1 + b_2 + \cdots + b_k + r_{(k+1)} + \cdots + r_n = r, \tag{2.2}$$

$$b_1 + 2b_2 + \cdots + kb_k + (k + 1)r_{(k+1)} + \cdots + nr_n = n. \tag{2.3}$$