NOTES

This section is devoted to brief research and expository articles, notes on methodology and other short items.

A NOTE ON THE THEORY OF MOMENT GENERATING FUNCTIONS

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Let X be a one-dimensional variate and let $F(x)$ be its distribution function.[1] The function

$$G(\alpha) = E(e^{\alpha X}) = \int_{-\infty}^{+\infty} e^{\alpha x}\, dF(x), \qquad \alpha \text{ real},$$

in which the integral is assumed to converge for α in some neighborhood of the origin, is called the moment generating function of X. In dealing with certain distribution problems, this function has been widely used by statisticians, and especially by the English writers, in place of the closely-related characteristic function $f(t) = E(e^{itX})$. It is known that a characteristic function uniquely determines the corresponding distribution, and that if a sequence of characteristic functions approaches a limit, the corresponding sequence of distribution functions does likewise. (These results are more accurately stated below.) The appropriate analogues for the moment generating function of these theorems are apparently not too readily accessible in the literature, if they have been treated at all, and it seems worthwhile to record them in this note.

Henceforth we abbreviate distribution function to d.f., moment generating function to m.g.f., and characteristic function to c.f. The variables α and t will always be real, in contradistinction to the complex variable s, to be introduced in the next paragraph.

The uniqueness property of the c.f. may be stated as follows: If $F_1(x)$ and $f_1(t)$ are the d.f. and c.f. of one variate, and $F_2(x)$ and $f_2(t)$ are those of another, and if $f_1(t) \equiv f_2(t)$ for all[2] t, then $F_1(x) \equiv F_2(x)$ for all x [1, p. 28]. To study the corresponding situation for the m.g.f., we first observe that

$$\varphi(s) = E(e^{sX}) = \int_{-\infty}^{+\infty} e^{sx}\, dF(x), \qquad s \text{ complex},$$

[1] Or cumulative frequency function; our notation and terminology are uniform with that of [1] except for the use of the term "variate" instead of "random variable."

[2] It is possible for two non-identical distributions to have c.f.'s which are identical throughout an interval of values of t containing the origin; an example is given in [4], p. 190. The author is obliged to Professor Wintner and Professor Feller for pointing out the existence of this particular example.