A METHOD OF DETERMINING EXPLICITLY THE COEFFICIENTS OF THE CHARACTERISTIC EQUATION

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1. Introduction. When an investigator is interested in all of the latent roots of the characteristic equation of a matrix and not in its latent vectors, it is sometimes desirable to expand out the determinental equation in order to determine explicitly the polynomial coefficients $(p_1, p_2, \cdots, p_n)$ in the expression

(1) $$D(\lambda) = |\lambda I - a| = \lambda^n + p_1\lambda^{n-1} + \cdots + p_{n-1}\lambda + p_n .$$

This can be done in a variety of ways, all of which are necessarily somewhat tedious for high order matrices. Except for sign the coefficients are respectively the sum of a's principal minors of a given order. These can be computed efficiently by "pivotal" methods [1]. Alternatively through the utilization of the Cayley-Hamilton theorem, whereby a matrix satisfies its own characteristic equation, the p's appear as the solution of n linear equations [2, 3]. In a third method Horst has employed Newton's formula concerning the powers of roots to derive the p's as the solution of a triangular set of equations, the coefficients of the latter only being attained after considerable matrix multiplication [4]. A fourth method suggested to me by Professor E. Bright Wilson, Jr. of Harvard University, consists of evaluating $D(\lambda)$ for n values of λ, presumably by efficient "Doolittle" methods; to these n points, Lagrange's interpolation formula is applied to determine the n coefficients explicitly.

2. The New Method. The present paper describes a new computational method based upon well-known dynamical considerations. A single nth order differential equation can be converted into "normal" form, involving n first order differential equations. This is easily done by defining appropriate new variables. If the original nth order differential equation is written as

(2) $$X^{(n)}(t) + p_1X^{(n-1)}(t) + \cdots + p_{n-1}X'(t) + p_n = 0,$$

then the new normal system can be written as

(3) $$X_i'(t) = \sum_1^n b_{ij}X_j(t), \qquad (i = 1, \cdots n)$$

where

(4) $$[b_{ij}] = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 \\ -p_n & -p_{n-1} & -p_{n-2} & \cdots & -p_1 \end{bmatrix}$$

is the so-called companion matrix to the polynomial in question.