ASYMPTOTIC FORMULAS FOR SIGNIFICANCE LEVELS OF CERTAIN DISTRIBUTIONS

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1. Introduction. The purpose of this paper is to derive asymptotic formulas for the significance levels, or per cent points, of certain well-known statistical distributions.[1] Although we restrict ourselves here to two distributions, those of Chi-Square and of Student's t, it will be apparent that the methods used are applicable to many other distributions as well.

The following results are obtained. Let y_p be the p per cent point of the normal distribution, that is, the distribution defined by

$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}v^2}\, dv, \tag{1.1}$$

so that

$$\Phi(y_p) = 1 - p. \tag{1.2}$$

If $\chi^2_{p,n}$ and $t_{p,n}$ denote the p per cent points of the Chi-Square and Student's t distributions with n degrees of freedom respectively, then

$$\chi^2_{p,n} = n + y_p\sqrt{2n} + \frac{2}{3}(y_p^2 - 1) + \frac{y_p^3 - 7y_p}{9\sqrt{2n}} + o\left(\frac{1}{\sqrt{n}}\right), \tag{1.3}$$

and

$$t_{p,n} = y_p + \frac{y_p^3 + y_p}{4n} + o\left(\frac{1}{n}\right). \tag{1.4}$$

These formulas approximate the true values of $\chi^2_{p,n}$ and $t_{p,n}$ to a high degree of accuracy. Tables of comparative values for several values of p and n are given in Section 4.

We shall need the following theorem due to Cramér [3, p. 81; see also pp. 86–87].

Theorem 1: *Let $X_1, X_2, \cdots$ be a sequence of independent, identically distributed random variables having an absolutely continuous distribution function with mean value zero, dispersion σ and finite fifth absolute moment. Let $H_n(x)$ be the distribution function of $(X_1 + \cdots + X_n)/(\sigma\sqrt{n})$, and let $n^{-\frac{1}{2}(r-2)}\lambda_r$ denote the r-th semi-invariant of $H_n(x)$. Then*

$$\Phi(x) - H_n(x) = \frac{\lambda_3}{3!\sqrt{n}}\Phi^{(3)}(x) - \frac{\lambda_4}{4!\,n}\Phi^{(4)}(x) - \frac{10\lambda_3^2}{6!\,n}\Phi^{(6)}(x) + 0(n^{-3/2}). \tag{1.5}$$

[1] This problem was proposed to the author by J. H. Curtiss.