# AN EXTENSION OF WILKS' METHOD FOR SETTING TOLERANCE LIMITS

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**1. Introduction.** Let $x$ be a random variable and let $f(x)$ be its probability density function. Suppose that nothing is known about $f(x)$ except that it is continuous. Let $x_1, \cdots, x_n$ be $n$ independent observations on $x$. The problem of setting tolerance limits can be formulated as follows: *For some given positive values* $\beta < 1$ and $\gamma < 1$ *we have to construct two functions* $L(x_1, \cdots, x_n)$ *and* $M(x_1, \cdots, x_n)$, *called tolerance limits, such that the probability that*

$$\int_L^M f(t)\, dt \geq \gamma, \tag{1}$$

*holds, is equal to* $\beta$. This problem has recently been solved by S. S. Wilks[1] in a very satisfactory way when nothing is known about $f(x)$ except that it is continuous. Wilks proposes the following solution: Let $x_1, \cdots, x_n$ be the observed values of $x$ arranged in order of increasing magnitude. Then $L = x_r$ and $M = x_{n-r+1}$ where $r$ denotes a positive integer. The exact sampling distribution of the statistic $\int_{x_r}^{x_{n-r+1}} f(t)\, dt$ is derived by Wilks and this provides the solution for the problem of setting tolerance limits. A very important feature of Wilks' solution is the fact that the distribution of $\int_{x_r}^{x_{n-r+1}} f(t)\, dt$ is entirely independent of the unknown density function $f(x)$, i.e. the distribution of $\int_{x_r}^{x_{n-r+1}} f(t)\, dt$ is the same for any arbitrary continuous density function $f(x)$.

In this paper we shall give an extension of Wilks' method to the multivariate case. Let $x_1, \cdots, x_p$ be a set of $p$ random variables with the joint probability density function $f(x_1, \cdots, x_p)$. Suppose that nothing is known about $f(x_1, \cdots, x_p)$ except that it is a continuous function of $x_1, \cdots, x_p$. A sample of $n$ independent observations is drawn and the $\alpha$-th observation on $x_i$ is denoted by $x_{i\alpha}$ $(i = 1, \cdots, p; \alpha = 1, \cdots, n)$. The problem of setting tolerance limits for $x_1, \cdots, x_p$ can be formulated as follows: *For some given positive values* $\beta < 1$ *and* $\gamma < 1$ *we have to construct* $p$ *pairs of functions of the observations* $L_i(x_{11}, \cdots, x_{pn})$ *and* $M_i(x_{11}, \cdots, x_{pn})$ $(i = 1, \cdots, p)$ *such that the probability that*

$$\int_{L_p}^{M_p} \cdots \int_{L_1}^{M_1} f(t_1, \cdots, t_p)\, dt_1 \cdots dt_p \geq \gamma, \tag{2}$$

[1] S. S. Wilks, "Determination of sample sizes for setting tolerance limits," *Annals of Math. Stat.*, Vol. 12 (1941).