# ON SOLUTIONS OF THE BEHRENS-FISHER PROBLEM, BASED ON THE $t$-DISTRIBUTION

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**1. The Problem.** The problem [1, 2] is the interval estimation[1] of the difference of the means of two normal populations when the ratio of the variances of the populations is unknown. The reader who wishes to see the present solution before considering theoretical details will find it recapitulated in the Summary at the end and will want to refer to the following notation: $(x_1, x_2, \cdots, x_m)$ and $(y_1, y_2, \cdots, y_n)$ are random samples from normal populations with means $\alpha$ and $\beta$, and variances $\mu$ and $\nu$, respectively. Define $\delta = \alpha - \beta$. We assume $m \leqq n$, and that the variates in each sample are in the order of observation, or else have been randomized.

Recently Neyman [3] has called attention to a solution which we shall designate as (B), and which is a special case of an unpublished solution of Bartlett[2]. It will be simpler to describe (B) later, but we mention now that it has the following advantages: (*i*) its validity does not depend on the values of unknown parameters, (*ii*) the required computations are simple, and (*iii*) only existing tables are needed,——the widely available Fisher $t$-tables. An unsatisfactory aspect of (B) is that when the sample sizes are unequal, $n - m$ of the variates $y_i$ are completely discarded. The solution below shares with (B) the advantages (*i*), (*ii*), (*iii*); indeed, it is identical with (B) when $n = m$, but when $n \neq m$ it is free from the above objection.

**2. Simple Solution.** We begin with a simple restricted approach; later we will review the result from a somewhat broader standpoint. If random variables $d_1, d_2, \cdots, d_m$ are independently normally distributed with mean $\delta$ and variance $\sigma^2$, and if $L$ and $Q$ are defined from

$$L = \sum_{i=1}^{m} d_i/m, \qquad Q = \sum_{i=1}^{m} (d_i - L)^2,$$

then $m^{\frac{1}{2}}(L - \delta)/\sigma$ and $Q/\sigma^2$ are independently distributed; the former is a normal variable with zero mean and unit variance; the latter, $Q/\sigma^2 = \chi^2_{m-1}$, where $\chi^2_k$ is a generic notation for a random variable distributed according to the $\chi^2$-law with $k$ degrees of freedom. The quotient

[1] We treat the problem from the standpoint of confidence intervals, rather than significance tests, since when the former are available for $\delta$ so is a whole class of the latter, namely for any hypothesis $\delta = \delta_0$, for all $\delta_0$. Furthermore, questions of the existence of "best" tests and "best" confidence intervals are closely related [5a].

[2] How far Bartlett followed the path of this paper is not clear from the brief mention of his results by Welch [4], except that he did establish the sufficiency of certain orthogonality conditions.