NOTES

This section is devoted to brief research and expository articles, notes on methodology and other short items.

NOTE ON RUNS OF CONSECUTIVE ELEMENTS

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In my paper [1] I did not derive the asymptotic distribution of $W(R)$, an omission which I wish to correct in this note.

Let the stochastic variable $R = (x_1, \cdots, x_n)$ be a permutation of the first n positive integers, where each permutation has the same probability $\frac{1}{n!}$. A subsequence $x_{i+1}, x_{i+2}, \cdots, x_{i+l}$, is called a run of consecutive elements of length l if:

a) when l' is any integer such that $1 \leq l' < l$,

$$|x_{i+l'} - x_{i+l'+1}| = 1$$

b) when $i > 0$, $|x_i - x_{i+1}| > 1$

c) when $i + l < n$, $|x_{i+l} - x_{i+l+1}| > 1$.

Let $W(R)$ be the total number of runs in R. Then $n - W(R)$ is a stochastic variable which, it will be shown, has in the limit the Poisson distribution with mean value 2. More precisely, if $p(w)$ is the probability that $n - W(R) = w$, then

$$\lim_{n\to\infty} p(w) = \frac{2^w}{e^2 \cdot w!}. \tag{1}$$

Proof: Define stochastic variables $y_i(i = 1, 2, \cdots, n)$, as follows: $y_i = 1$ if x_i is the first element of a run of length 2, $y_i = 0$ otherwise. It is easy to see that the probability that $x_i(i = 1, 2, \cdots, n)$ be the initial element of a run of length greater than two is $O\left(\frac{1}{n^2}\right)$ and hence that the probability of the occurrence of a run of length greater than two is $O\left(\frac{1}{n}\right)$. Hence the limiting distribution of $n - W(R)$ is the same as that of

$$y = \sum_{i=1}^{n} y_i,$$

provided either exists.

The y_i are dependent stochastic variables and almost all (i.e., all with the exception of a fixed number) have the same marginal distribution. We now wish to consider the expression

$$E(y_{i_1}^{\alpha_1} y_{i_2}^{\alpha_2} \cdots y_{i_k}^{\alpha_k})$$

(where the symbol E denotes the expectation) for any set of fixed positive integers k, $\alpha_1, \cdots, \alpha_k$, and for all k-tuples $i_1, i_2, \cdots, i_k$, with no two elements