ON THE DISTRIBUTION OF THE RADIAL STANDARD DEVIATION

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1. Introduction. Of interest in the field of ballistics is a measure of the accuracy of bullets. In acceptance tests of small arms ammunition lots, for example, a sample of rounds from each lot is fired from a fixed rifle at a vertical target placed a specified distance from the rifle. The accuracy of the bullets is taken to be some measure of the scattering (or lack of scattering) of the bullet holes on the target. The purpose of such a test would be to determine whether or not the lot under consideration differs significantly in accuracy from (a) standard values or (b) its predecessors.

One useful measure of accuracy is the radial standard deviation which is defined by the relation

$$Z = \sqrt{\frac{1}{N}\{\Sigma(x_i - \bar{x})^2 + \Sigma(y_i - \bar{y})^2\}}, \tag{1}$$

where x_i and y_i are respectively the abscissa and ordinate of any point measured from an arbitrary origin and N is the sample size.

It will be the purpose of the present discussion to call attention to a series expansion for the distribution of the statistic Z in samples of N assuming that the distribution of all rounds of the lot on the target follow the bivariate normal population law

$$f(x, y) = \frac{1}{2\pi\sigma_1\sigma_2} e^{-\frac{x^2}{2\sigma_1^2}-\frac{y^2}{2\sigma_2^2}}, \qquad (x \text{ and } y \text{ statistically independent}) \tag{2}$$

where σ_1^2 and σ_2^2 are the parent variances of x and y respectively. In the above probability density function, the population means are taken to be zero since the statistic Z is quite independent of the origin selected.

2. Moment generating function of Z^2. The distribution of $s_1^2 = \frac{1}{N}\Sigma(x_i - \bar{x})^2$ in samples of N from a normal population is given by the well-known law,

$$dF(s_1^2) = \frac{\frac{N}{2\sigma_1^2}}{\Gamma\left(\frac{N-1}{2}\right)}\left(\frac{Ns_1^2}{2\sigma_1^2}\right)^{\frac{1}{2}(N-3)} e^{-\frac{Ns_1^2}{2\sigma_1^2}}\, ds_1^2, \qquad s_1^2 \geq 0. \tag{3}$$

The moment generating function of s_1^2 may be found (in a neighborhood of $t = 0$) by straightforward integration:

$$M_{s_1^2}(t) = E(e^{s_1^2 t}) = \int_0^\infty e^{s_1^2 t}\, dF(s_1^2) = \left\{1 - \frac{2\sigma_1^2 t}{N}\right\}^{-\frac{1}{2}(N-1)} \tag{4}$$

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