NOTES

This section is devoted to brief research and expository articles, notes on methodology and other short items.

ON DISTRIBUTION-FREE TOLERANCE LIMITS IN RANDOM SAMPLING

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Let $X_1, \cdots, X_n$ be independent random variables each with the continuous and differentiable cumulative distribution function $\sigma(x) = Pr(X_i < x)$. A continuous function $f(x_1, \cdots, x_n)$ with the property that the random variable $Y = \sigma(f(X_1, \cdots, X_n))$ has a probability distribution which is independent of $\sigma(x)$ will be called a *distribution-free upper tolerance limit*[1] (d. f. u. t. l.). We shall prove

THEOREM 1. *A necessary and sufficient condition that the continuous function $f(x_1, \cdots, x_n)$ be a d. f. u. t. l. is that the function*

$$\bar{f}(x_1, \cdots, x_n) = \prod_{i=1}^{n} \{f(x_1, \cdots, x_n) - x_i\}$$

be identically zero.

PROOF. Since f is continuous, we can prove the necessity of the condition by deriving a contradiction from the assumption that f is a d. f. u. t. l. for which there exist distinct numbers $a_1, \cdots, a_n$ such that $f(a_1, \cdots, a_n) = A \neq a_i$, $(i = 1, \cdots, n)$.

Since the numbers $a_1, \cdots, a_n, A$ are distinct, there will exist a positive number ϵ such that the $(n + 1)$ intervals

$$I: \quad A - \epsilon \leq x \leq A + \epsilon$$

$$I_i: \quad a_i - \epsilon \leq x \leq a_i + \epsilon \qquad (i = 1, \cdots, n),$$

have no points in common. Moreover, since f is continuous, there will correspond to ϵ a positive number $\epsilon_1 < \epsilon$ such that

$$A - \epsilon \leq f(x_1, \cdots, x_n) \leq A + \epsilon,$$

provided that simultaneously

$$|x_i - a_i| < \epsilon_1 \qquad (i = 1, \cdots, n).$$

Now let p be any number between $\frac{1}{3}$ and $\frac{2}{3}$. Corresponding to p we define the function $\sigma_p(x)$ as follows. In the interval I we set $\sigma_p(x) = p$. In every interval

$$J_i: \quad a_i - \epsilon_1 \leq x \leq a_i + \epsilon_1 \qquad (i = 1, \cdots, n)$$

[1] Cf. S. S. Wilks, *Mathematical Statistics*, Princeton University Press (1943), pp. 93–94.