THE PROBABILITY OF CONVERGENCE OF AN ITERATIVE PROCESS OF INVERTING A MATRIX

By Joseph Ullman

Columbia University

Introduction. The inversion of a matrix is a computational problem of wide application. This is a further study of an efficient iterative method of matrix inversion described by Harold Hotelling [1], with an examination of the probability of convergence in relation to the accuracy of the initial approximation. The lines of investigation were suggested both by his article and by helpful comments made during the course of the research.

The inverse of a matrix can be obtained to any desired degree of accuracy by using a variation of the Doolittle method, and starting with a sufficient number of accurate decimal places in the matrix being inverted. This procedure becomes inefficient if the order of the matrix is large, or if the desired degree of accuracy is very great. In either case the efficiency can be greatly increased by first obtaining an approximation to a small number of decimal places and then applying a method of iteration until the desired accuracy is achieved.

1. Iterative methods. Hotelling's method of iteration is as follows. Let A be the matrix to be inverted and let C_0 be the approximation to the inverse. Calculate in turn $C_1, C_2, \cdots$ where,

$$C_{m+1} = C_m(2 - AC_m). \tag{1.1}$$

This sequence of matrices will converge to the inverse of A if the roots of

$$D = 1 - AC_0, \tag{1.2}$$

are all less than unity in absolute value.

The iterative method (1.1) will be generalized to yield a class of iterative methods, one element of which will be shown to be more efficient, in certain cases, than method (1.1). The generalized iterative method is,

$$C_{m+1} = C_m\{1 + (1 - AC_m) + (1 - AC_m)^2 + \cdots + (1 - AC_m)^{k-1}\}. \tag{1.3}$$

For every k, the condition for convergence is that the roots of the matrix (1.2) all be less than unity in absolute value.

A method of comparing the efficiency of these different iterative methods arises from the following considerations. Since

$$C_0 = A^{-1}(AC_0), \tag{1.4}$$

which is equivalent to

$$C_0 = A^{-1}(1 - D), \tag{1.5}$$