RANDOM ALMS

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1. Statement of the problem. Consider the problem of distributing one pound of gold dust at random among a countably infinite set of beggars. Let the beggars be enumerated and let the procedure for distribution be as follows: the first beggar is given a random portion of the gold; the second beggar gets a random portion of the remainder; $\cdots$ and so on ad infinitum. In this description the phrase "random portion" occurs an infinite number of times: it seems reasonable to require that it have the same interpretation each time. To be precise: let x_j ($j = 0, 1, 2, \cdots$) be the amount received by the jth beggar. Let the distribution of x_0 be given by a density function $p(\lambda)$:

$$p(\lambda) \geqq 0, \qquad 0 \leqq \lambda \leqq 1; \tag{1}$$

$$\int_0^1 p(\lambda)\, d\lambda = 1; \tag{2}$$

$$P(a < x_0 < b) = \int_a^b p(\lambda)\, d\lambda, \qquad 0 \leqq a < b \leqq 1. \tag{3}$$

After the first beggar has received his alms and the amount of gold dust left is μ, (i.e. $x_0 = 1 - \mu$), the value of x_1 will be between 0 and μ. The uniformity requirement mentioned above means that the proportion of μ that the second beggar is to receive is again determined by the probability density p: in other words the conditional probability that x_1 be between $\lambda\mu$ and $(\lambda + d\lambda)\mu$, given that $x_0 = 1 - \mu$, is $p(\lambda)\, d\lambda$. In symbols:

$$P(a\mu < x_1 < b\mu \mid x_0 = 1 - \mu) = \int_a^b p(\lambda)\, d\lambda. \tag{4}$$

Writing $\alpha = \alpha\mu$, $\beta = b\mu$, (4) becomes

$$P(\alpha < x_1 < \beta \mid x_0 = 1 - \mu) = \int_\alpha^\beta \frac{1}{\mu}\, p\left(\frac{\lambda}{\mu}\right) d\lambda. \tag{5}$$

More generally I shall assume that the conditional probability distribution of x_n, assuming that after the preceding donations there is left an amount μ, is given in the interval $(0, \mu)$ by $\frac{1}{\mu}\, p\left(\frac{\lambda}{\mu}\right)$. In symbols:

$$P\Big(a < x_n < b \,\Big|\, \sum_{j<n} x_j = 1 - \mu\Big) = \int_a^b \frac{1}{\mu}\, p\left(\frac{\lambda}{\mu}\right) d\lambda, \qquad 0 \leqq a < b \leqq \mu. \tag{6}$$

This assumption completely determines (in terms of p) the joint distribution of the whole infinite sequence $\{x_0, x_1, x_2, \cdots\}$. Several interesting special