STATISTICAL ANALYSIS OF CERTAIN TYPES OF RANDOM FUNCTIONS

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1. Introduction. In solving certain physical problems (Brownian movements, shot effect) one is often led to the study of superpositions of random pulses. More precisely, one is led to sums of the type

$$F(t) = \sum_{j=1}^{N} f(t - t_j), \tag{1}$$

where N and the t_j's are random variables and a function $P(t)$ is given such that $\int_\Delta P(t)\,dt$ represents the average number of pulses occurring during the time interval Δ.

We propose to give a fairly detailed treatment of those statistical properties of $F(t)$ which may be of interest to a physicist and at the same time pay careful attention to the mathematical assumptions which underly the applications. It may also be pointed out that our results could be applied to the theory of time series.

2. Statistical assumptions and the distribution of N. The statistical assumptions can be formulated as follows:

1. The t_j's form an infinite sequence of independent identically distributed random variables each having $p(t)$ as its probability density.
2. N is capable of assuming the values $0, 1, 2, 3, \cdots$ only, and N is independent of the t_j's.
3. If $M(\Delta; N)$ denotes the number of those t_j's among the first N, which fall within the interval Δ, then for non-overlaping intervals Δ_1 and Δ_2 the random variables $M(\Delta_1; N)$ and $M(\Delta_2; N)$ are independent.

We now state our first theorem.[1]

Theorem 1. *Assumptions* 1, 2, 3 *imply that N is distributed according to Poisson's law, i.e.*

$$\text{Prob}\,\{N = r\} = e^{-h}\frac{h^r}{r!},$$

where $h = \int_{-\infty}^{+\infty} P(t)\,dt$.

[1] For a different approach to Poisson's distribution see W. Feller, *Math. Ann.* 113 (1937) in particular pp. 113–160.