ON CUMULATIVE SUMS OF RANDOM VARIABLES

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1. Introduction. Let $\{z_i\}$ (i = 1, 2, $\cdots$, ad inf.) be a sequence of independent random variables each having the same distribution. Denote by Z_j the sum of the first j elements of the sequence $\{Z_i\}$, i.e.,

$$Z_j = z_1 + z_2 + \cdots + z_j \quad (j = 1, 2, \cdots, \text{ad inf.}). \tag{1}$$

Let a be a given positive constant and b a given negative constant. Denote by n the smallest positive integer for which Z_n lies outside the open interval (b, a), i.e., Z_n is either $\leq b$ or $\geq a$. Obviously n is a random variable. If $b < Z_i < a$ for $i = 1, 2, \cdots$, ad inf., we shall say that $n = \infty$.

For any relation R we shall denote the probability that R holds by $P(R)$. It will be shown later that $P(n = \infty) = 0$, provided the variance of z_i is positive.

In this paper we shall deal with the problem of obtaining the value of $P(Z_n \geq a)$[1] and that of finding the probability distribution of n.

The study of such cumulative sums is of interest in various statistical problems. For example, a multiple sampling scheme proposed recently by Walter Bartky[2] makes use of such cumulative sums.

Cumulative sums also play an important role in the theory of the random walk of interest in physics. The results obtained in this paper may have bearing particularly on the theory of the random walk with absorbing barriers. In the presence of an absorbing wall the random walk stops whenever the particle arrives at the wall, i.e., whenever the cumulative sum of the displacements reaches a certain value.[3]

2. Two Lemmas. Lemma 1. *If the variance of z_i is not zero, $P(n = \infty) = 0$.*

Proof: Let $c = |a| + |b|$. If $n = \infty$ then for any positive integer r the following inequalities must hold

$$\left(\sum_{i=kr+1}^{(k+1)r} z_i\right)^2 < c^2 \quad (k = 0, 1, 2, \cdots, \text{ad inf.}). \tag{2}$$

To prove $P(n = \infty) = 0$, it is sufficient to show that the probability is zero that (2) holds for all integer values of k. Since the variance of z_i is not zero, the ex-

[1] Since $P(n = \infty) = 0$, we have $P(Z_n \leqq b) = 1 - P(Z_n \geqq a)$.

[2] "Multiple sampling with constant probability", *Annals of Math. Stat.*, Vol. 14 (1943), pp. 363–377.

[3] See in this connection S. Chandrasekhar, "Stochastic problems in physics and astronomy", *Rev. of Modern Physics*, Vol. 15 (1943), p. 5.