RANGES AND MIDRANGES[1]

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1. Introduction. In the following the generating functions of the extremes are studied in order to determine the nature of the distributions of the ranges and the midranges.

A large sample of size n is considered to be drawn from an unlimited symmetrical continuous distribution with zero mean. The difference and the sum of the largest and of the smallest observation, the extremes, are called *range* and *midrange*. R. A. Fisher and L. H. C. Tippett [2] have established the limiting distributions of the largest and of the smallest member of a sample. The exact conditions under which these distributions hold have been given by R. von Mises [4]. For a normal distribution, L. H. C. Tippett [7] has calculated the numerical values of the mean range and the first four moments of the range for sample sizes varying from 2 to 1000. He has shown that, for sample sizes exceeding 200, the correlation between the largest and the smallest observation may be neglected. Later, E. S. Pearson [5] has calculated the probability function of the range for small samples ($n = 2$ to 20) taken from a normal population. These calculations are very laborious. Recently, W. E. Deming [1] has applied the range to quality control.

The concepts "extremes", "range" and "midrange" allow a simple generalization. Let ${}_mx$ and x_m be the mth observation in increasing and in decreasing magnitude, henceforth called mth value "from below" and "from above". As long as the index m is small compared to the sample size n, the mth values under consideration are extremes. The difference and the sum of the mth extreme observations are called the mth *range* and the mth *midrange*. We will investigate the asymptotic distributions of the mth extremes, of the mth range, and of the mth midrange. Assuming that the number of observations is very large, the correlation between the largest and the smallest observation may be neglected. Then the mth range and the mth midrange are the difference and the sum of two independent variates, the mth extremes.

It was found that the distribution of the mth range is skew and the distribution of the mth midrange is of the generalized logistic type, which is symmetrical. For m increasing the distributions of the mth extremes, the mth ranges, and the mth midranges converge toward normality.

2. Generating functions of the *m*th extremes. Let $\varphi(x)$ be an initial continuous symmetrical distribution with mean zero; let u_m be the most probable mth value from above; let α_m be defined by

$$\alpha_m = \frac{n}{m}\,\varphi(u_m). \tag{1}$$

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