THE EXPECTED VALUE AND VARIANCE OF THE RECIPROCAL AND OTHER NEGATIVE POWERS OF A POSITIVE BERNOULLIAN VARIATE[1]

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1. Introduction. The expected value of the reciprocal of a Bernoullian variate appears in certain problems of random sampling wherein both practical considerations and mathematical necessity make zero an inadmissible value of the variate. This special condition excluding zero is necessary from a practical standpoint because statistics can not be calculated from an empty class. It is a necessary condition, in the mathematical sense, for the expected value, and variances involving it, to be finite. When subject to this condition the Bernoullian variate will be designated the *positive* Bernoullian variate.

There appears to be no simple expression for the expected value of the reciprocal such as there is for the expected value of positive integral powers of the positive Bernoullian variate. This paper presents in (15) a factorial series, which can be computed conveniently to any desired number of terms by means of the recursion relation (18). Upper and lower bounds on the remainder may be computed readily from (20), (21), (23), (24), and (26) and the approximation may be improved by adding an estimate of the remainder taken between these bounds. A factorial series for the expected value of negative integral powers is given in (34). A factorial series for the expected value of the reciprocal of the positive hypergeometric variate is given in (53). Series for the variances follow directly from the series for expected values.

A simple example of the sampling problems in which this expected value appears is presented by the following instance of estimates derived from samples of variable size:

An infinite population consists of items of two kinds or classes, A and B. Lots of N items each are drawn at random. In such lots the number of items, x', that are of class A is an ordinary Bernoullian variate. Next, every lot composed entirely of items of class B is discarded. This excludes all lots for which $x' = 0$. From each remaining lot the $N - x'$ items of class B are set aside, leaving a sample composed entirely of items of class A. The number of such items, x, varies from sample to sample. It will be designated a positive Bernoullian variate since $x = x'$ if $x' > 0$ and x does not exist if $x' \leq 0$. Finally, let there be associated with each item in class A a particular value of a variable, y, the variance of which in A is σ^2. Then if the mean value of y is computed for each sample, the error variance of such means is $E(\sigma^2/x) = \sigma^2 E(1/x)$.

Instances similar to that just described occur in the design of sampling surveys from which statistics are to be obtained separately for each of several classes

[1] Developed from a section of a paper presented to the Washington meeting of the Institute of Mathematical Statistics on June 18, 1943.