THE APPROXIMATE DISTRIBUTIONS OF THE MEAN AND VARIANCE OF A SAMPLE OF INDEPENDENT VARIABLES

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1. Introduction. In this paper we shall study the mean and variance of a large number, n (a sample of size n) of mutually independent random variables:

$$(1) \qquad \xi_1, \xi_2, \cdots, \xi_n,$$

having the same probability distribution represented by a (cumulative) distribution function $P(x)$. The rth moment, absolute moment, and semi-invariant of $P(x)$ are denoted by α_r, β_r, and γ_r respectively. It is assumed that for a certain integer $k \geq 3$, $\beta_k < \infty$ and that $\alpha_2 > 0$. Hence there is no loss of generality in assuming that

$$(2) \qquad \alpha_1 = 0, \qquad \alpha_2 = 1.$$

The characteristic function corresponding to $P(x)$ is denoted by $p(t)$.

We put

$$(3) \qquad \bar{\xi} = \frac{1}{n}\sum_{r=1}^{n} \xi_r, \qquad \eta = \frac{1}{n}\sum_{r=1}^{n} (\xi_r - \bar{\xi})^2$$

$$(4) \qquad F(x) = Pr\{\sqrt{n}\,\bar{\xi} \leq x\}, \qquad G(x) = Pr\left\{\frac{\sqrt{n}(\eta - 1)}{\sqrt{\alpha_4 - 1}} \leq x\right\}.$$

The definition of $G(x)$ implies that $\alpha_4 < \infty$ and $\alpha_4 - 1 > 0$. The case $\alpha_4 - 1 = 0$ provides an easy degenerated case which will be treated separately (section 4).

Cramér's theorem of asymptotic expansion[1] reads as follows:

Theorem 1. *If $P(x)$ is non-singular and if $\beta_k < \infty$ for some integer $k \geq 3$, then*

$$(5) \qquad F(x) = \Phi(x) + \Psi(x) + R(x)$$

where

$$(6) \qquad \Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}y^2}\, dy.$$

$\Psi(x)$ is a certain linear combination of successive derivatives $\Phi^{(3)}(x), \cdots, \Phi^{(3(k-3))}(x)$ with each coefficient of the form $n^{-\frac{1}{2}\nu}$ times a quantity depending only on k, $\alpha_3, \cdots, \alpha_{k-1}$ $(1 \leq \nu \leq k-3)$ and

$$(7) \qquad |R(x)| \leq Q/n^{\frac{1}{2}(k-2)}$$

where Q is a constant depending only on k and $P(x)$.

[1] H. Cramér: *Random Variables and Probability Distributions* (1937), Ch. 7. This book will be referred to as (C).