ON THE DISTRIBUTION OF THE SERIAL CORRELATION COEFFICIENT

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The distribution of the serial correlation coefficient, in samples drawn from a parent distribution with zero serial correlation, has been studied by many authors. Anderson [1] obtained the exact distribution. Dixon [3] and Koopmans [4] have given approximate distributions, each attained by smoothing the characteristic values of the numerator of $\bar{r}$ in (1) below. Dixon smoothed the characteristic values in the generating function and obtained his results by comparing the moments of the exact distribution with those of the approximation, of which the first T are found to be exact. Koopmans smoothed the characteristic values in the exact distribution function. Here we evaluate Koopmans result and show that it is the same as Dixon's approximation. It thus appears that in this case it is immaterial whether the characteristic values are smoothed before or after inverting the characteristic function. We also add Tables comparing confidence limits for the exact distribution, for the approximation referred to, and for a normal approximation.

We define the serial correlation coefficient as

$$\bar{r} = \frac{\sum_{t=1}^{T} x_t x_{t+1}}{\sum_{t=1}^{T} x_t^2}, \quad x_{T+1} = x_1. \tag{1}$$

Then Koopmans obtains, if the true value ρ of $\bar{r}$ equals 0, and the x_t are normally and independently distributed with mean 0 and variance σ^2, the approximate distribution $T/2 - 2$.

$$\bar{h}(\bar{r}, T) = \frac{2^{\frac{1}{2}T}(\frac{1}{2}T - 1)}{\pi} \int_0^{\text{arc cos } \bar{r}} (\cos \alpha - \bar{r})^{\frac{1}{2}T-2} \sin \tfrac{1}{2}T\alpha \sin \alpha \, d\alpha. \tag{2}$$

Although in the distribution problem T is a positive integer, it is useful to consider the right-hand member of (2) as the definition of $\bar{h}(\bar{r}\cdot T)$ for those complex values of T for which it exists.

Let $R(T)$ denote the real part of T. If $R(T) > 2N + 2$, we obtain

$$\begin{aligned} \frac{d^T}{d\bar{r}^N} \bar{h}(\bar{r}, T) = {} & \frac{(-1)^N 2^{\frac{1}{2}T}(\frac{1}{2}T - 1)}{\pi} (\tfrac{1}{2}T - 2)(\tfrac{1}{2}T - 3) \cdots (\tfrac{1}{2}T - N - 1) \\ & \cdot \int_0^{\text{arc cos } \bar{r}} (\cos \alpha - \bar{r})^{\frac{1}{2}T-2-N} \sin \tfrac{1}{2}T\alpha \sin \alpha \, d\alpha. \end{aligned} \tag{3}$$

Now, according to [2], tables 41, 42.

$$\begin{aligned} & \int_0^{\pi/2} (\cos \alpha)^{\frac{1}{2}T-2-N} \sin \tfrac{1}{2}T\alpha \sin \alpha \, d\alpha \\ & = \frac{\frac{1}{2}T\pi}{2^{\frac{1}{2}T-N}} \frac{\Gamma(\frac{1}{2}T - N - 1)}{\Gamma(\frac{1}{2}(T - N + 1))\Gamma(\frac{1}{2}(1 - N))}. \end{aligned} \tag{4}$$