ON THE APPROXIMATE DISTRIBUTION OF RATIOS

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The purpose of this paper is to apply Cramer's theorem of asymptotic expansion[1] and Berry's theorem[2] to study the approximate distribution of ratios of the following two types:

$$\text{(I)} \qquad Z = \frac{1}{n}(Y_1 + \cdots + Y_n) \Big/ \frac{1}{m}(\bar{X}_1 + \cdots + \bar{X}_m) = \bar{Y}/\bar{X},$$

$$\text{(II)} \qquad Z = Y \Big/ \frac{1}{m}(X_1 + \cdots + X_m) = Y/\bar{X}.$$

In (I) the X_i, Y_j are independent, the Y_j are equi-distributed,[3] and the X_i are equi-distributed and positive. In (II) $X_1, \cdots, X_n$, Y are independent and positive, and the X_i are equi-distributed.

1. The ratio (I). Assume that (I1) the absolute kth moment of X_i and that of Y_j are finite and positive, where k is a fixed integer ≥ 3,
(I2) the distribution of X_i and that of Y_j are non-singular.

Let

$$\xi = \epsilon(X_i), \qquad \eta = \epsilon(Y_j), \qquad \sigma^2 = \epsilon(X_i^2) - \xi^2, \qquad \tau^2 = \epsilon(Y_j^2) - \eta^2$$

and

$$U = \frac{\sqrt{m}}{\sigma}(\bar{X} - \xi), \qquad V = \frac{\sqrt{n}}{\tau}(\bar{Y} - \eta).$$

Let $F(x)$, $G(x)$ and $H(x)$ be respectively the distribution functions of Z, U and V. Let

$$b = \left(\frac{\sigma^2 x^2}{m} + \frac{\tau^2}{n}\right)^{\frac{1}{2}}, \qquad u = \frac{\xi n - \eta}{b}.$$

Then the relation $Z \leq x$ is equivalent to

$$-\frac{x\sigma U}{b\sqrt{m}} + \frac{\tau V}{b\sqrt{n}} \leq u.$$

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[1] H. Cramér. *Random Variables and Probability Distributions* (1937), Chap. 7.

[2] A. C. Berry. "The accuracy of the Gaussian approximation to the sum of independent variates", *Trans. Amer. Math. Soc.*, Vol. 49 (1941), pp. 122–136.

[3] The Y_j are said to be equi-distributed if all Y_j have the same distribution function.