ON THE NORMAL APPROXIMATION TO THE BINOMIAL DISTRIBUTION

By W. Feller

Cornell University

1. Although the problem of an efficient estimation of the error in the normal approximation to the binomial distribution is classical, the many papers which are still being written on the subject show that not all pertinent questions have found a satisfactory solution. Let for a fixed n and $0 < p < 1$, $q = 1 - p$,

$$(1) \qquad T_k = \binom{n}{k} p^k q^{n-k}, \qquad P_{\lambda,\nu} = \sum_{k=\lambda}^{\nu} T_k .$$

For reasons of tradition (and, apparently, only for such reasons) one sets

$$(2) \qquad z_k = (k - np)\sigma^{-1}, \qquad \sigma = (npq)^{1/2},$$

and compares (1) with

$$(3) \qquad N_k = (2\pi)^{-1/2}\sigma^{-1}e^{-z_k^2/2} \quad \text{and} \quad \Pi_{\lambda,\nu} = \Phi\left(z_\nu + \frac{1}{2\sigma}\right) - \Phi\left(z_\lambda - \frac{1}{2\sigma}\right)$$

respectively,[1] where $\Phi(z)$ stands for the normalized error function. Many estimates are available for the maximum of the difference $| P_{\lambda,\nu} - \Pi_{\lambda,\nu} |$ for all λ, ν. Now this error is $O(\sigma^{-1})$ and even a precise appraisal will break down in the two most interesting cases: if σ is small, or if λ and ν are large as compared to σ. Indeed, even for moderately large values of k (such as are usually considered) the contribution of T_k to the sum in (1) will be considerably smaller than σ^{-1} so that any estimate of the form $O(\sigma^{-1})$ leaves us without guidance. With some modifications this remains true also for more refined estimates like Uspensky's remarkable result[2]

$$(4) \qquad P_{\lambda,\nu} = \Pi_{\lambda,\nu} + \frac{q - p}{6\sigma(2\pi)^{1/2}}\left[(1 - z^2)e^{-z^2/2}\right] \Big|_{z_\lambda - 1/2\sigma}^{z_\nu + 1/2\sigma} + \omega$$

with

$$| \omega | < \{.13 + .18 \,| p - q |\}\sigma^{-2} + e^{-3\sigma/2}$$

provided $\sigma \geq 5$. What is really needed in many applications is an estimate of the relative error, but this seems difficult to obtain.

It should also be noticed that the accuracy of the normal approximation to the binomial is by no means quite as good as many texts would make appear. Exam-

---

[1] Very often the limits z_λ and z_ν instead of $z_\nu + \frac{1}{2\sigma}$ and $z_\lambda - \frac{1}{2\sigma}$ are used. This naturally results in an unnecessary systematic undervaluation.

[2] Uspensky [3], p. 129. A two-term development of T_r with an error of $O(\sigma^{-2})$ valid for $| x | < 2$, $\sigma > 3$ has been given by Mirimanoff and Dovaz [1927].