NOTES

This section is devoted to brief research and expository articles, notes on methodology and other short items.

ESTIMATING THE PARAMETERS OF A RECTANGULAR DISTRIBUTION

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1. Introduction. In this note, the range and midrange of the sample are shown to be a pair of sufficient statistics, and maximum likelihood estimates, for the true range and true mean of a rectangular distribution; exact and limiting distribution of midrange, range, and their ratio are derived; the "efficiencies" of the sample mean and median as estimates of the true mean are calculated; and the limiting distribution of the difference between two sample midranges is derived. All the limiting distributions are non-normal, and the error of estimate is of order n^{-1} rather than the customary order $n^{-\frac{1}{2}}$. The limiting distribution of midrange, and the limiting ratio of variances of the midrange and sample mean were given by Fisher [1].

$f(x)$ and $F(x)$ are used throughout to designate the probability density function of x and the distribution function (cumulative probability function) of x; the argument will also indicate the random variable being considered.

2. Exact distribution of midrange, range, and their ratio. Let $x_1, \cdots, x_n$ be a set of n independent observations on a random variable having the rectangular distribution $f(x) = 1/L$, $(\theta - L/2 \leq x \leq \theta + L/2)$, where θ is the true mean, and L the true range. The minimum observation u and the maximum observation v are a pair of sufficient statistics for θ and L, as the conditional distribution of the remaining observations for given u and v is independent of θ and L:

$$f(x_1, \cdots, x_n \mid u, v) = (v - u)^{-(n-2)}$$

The midrange $\bar{\theta} = \frac{1}{2}(u + v)$ and the range $\bar{L} = v - u$ are maximum likelihood estimates of θ and L, respectively, as they are the parameter values which uniquely maximize $f(x_1, \cdots, x_n)$ for the given set of observations. We shall assume that the random variable is normalized by change of origin and change of scale so that $\theta = 0$ and $L = 1$. The joint probability density function of u and v is

$$f(u, v) = \frac{d^2 F(u, v)}{dv\, d(-u)} = \frac{d^2(v - u)^n}{dv\, d(-u)} \tag{1}$$
$$= n(n - 1)(v - u)^{n-2}, \qquad (-\tfrac{1}{2} \leq u \leq v \leq \tfrac{1}{2}).$$