APPROXIMATION OF THE DISTRIBUTION OF THE PRODUCT OF BETA VARIABLES BY A SINGLE BETA VARIABLE

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1. Introduction. In an article published elsewhere in the present issue of the *Annals of Mathematical Statistics* [1] the g-th moments of two statistical test criteria L_{mvc} and L_{vc} were found to have the following expressions, respectively:

$$(1)\qquad (k-1)^{g(k-1)}\prod_{i=1}^{k-1}\left[\frac{\Gamma(\frac{1}{2}(n-1-i)+g)}{\Gamma(\frac{1}{2}(n-1-i))}\right]\cdot\frac{\Gamma(\frac{1}{2}n(k-1))}{\Gamma(\frac{1}{2}n(k-1)+g(k-1))}$$

and

$$(2)\qquad (k-1)^{g(k-1)}\prod_{i=1}^{k-1}\left[\frac{\Gamma(\frac{1}{2}(n-1-i)+g)}{\Gamma(\frac{1}{2}(n-1-i))}\right]\cdot\frac{\Gamma(\frac{1}{2}(n-1)(k-1))}{\Gamma(\frac{1}{2}(n-1)(k-1)+g(k-1))}.$$

If we denote by $(a)_g$ the expression $a(a+1)(a+2)\cdots(a+g-1)$ and make use of the fact that

$$(3)\qquad \Gamma(a+g) = \Gamma(a)\cdot(a)_g$$

and

$$(4)\qquad \Gamma(a+rg) = \Gamma(a)\cdot(a)_{rg} = \Gamma(a)\cdot r^{rg}\prod_{i=1}^{r}\left(\frac{a+i-1}{r}\right)_g$$

where r is a positive integer, the two moments (1) and (2) reduce to

$$(5)\qquad \frac{\prod_{i=1}^{k-1}\left(\frac{n}{2}+\frac{i-k-1}{2}\right)_g}{\prod_{i=1}^{k-1}\left(\frac{n}{2}+\frac{i-1}{k-1}\right)_g}\quad\text{and}\quad\frac{\prod_{i=1}^{k-1}\left(\frac{n-1}{2}+\frac{i-k}{2}\right)_g}{\prod_{i=1}^{k-1}\left(\frac{n-1}{2}+\frac{i-1}{k-1}\right)_g}$$

respectively.

For any given value of i $(i = 1, 2, \cdots, k-1)$ the ratio

$$\frac{\left(\frac{n}{2}+\frac{i-k-1}{2}\right)_g}{\left(\frac{n}{2}+\frac{i-1}{k-1}\right)_g}\quad\text{or}\quad\frac{\left(\frac{n-1}{2}+\frac{i-k}{2}\right)_g}{\left(\frac{n-1}{2}+\frac{i-k}{k-1}\right)_g}$$

may be expressed in the form

$$\frac{\Gamma(p_i+g)}{\Gamma(p_i+q_i+g)}$$

which is the g-th moment of a beta variable u_i distributed according to

$$\frac{\Gamma(p_i+q_i)}{\Gamma(p_i)\Gamma(q_i)}\,u_i^{p_i-1}(1-u_i)^{q_i-1}\,du_i\,.$$