ON THE FIRST TWO MOMENTS OF THE MEASURE OF A RANDOM SET

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1. Introduction. In a recent paper [3] H. E. Robbins derived general formulas for the moments of the measure of any random set X, and applied the formulas to find the mean and the variance of a random sum of intervals on a line. In subsequent papers, J. Bronowski and J. Neyman [1], using other methods, found the variance when X is a random sum of rectangles in the plane, and H. E. Robbins [4] found the variance when X is a random sum of n-dimensional intervals in n-dimensional euclidean space. In the latter paper Robbins solved also the corresponding problem for circles on the plane.

Using the methods of Robbins, our purpose in the present paper is to solve the following similar problems:

(i) Let R denote the rectangle consisting of all points (x,y) such that $0 \leq x \leq A_1$, $0 \leq y \leq A_2$, and let R' denote the larger rectangle for which $-\delta \leq x \leq A_1 + \delta$, $-\delta \leq y \leq A_2 + \delta$. Let ρ denote a rectangle of fixed dimensions, $a \times b$, but variable position in the plane. The position of ρ will be determined by the coordinates x, y of its center P and the angle φ between the side of length a and the x-axis. We suppose $(a^2 + b^2)^{\frac{1}{2}} \leq \min(A_1, A_2, \delta)$. Let a fixed number N of rectangles ρ be chosen independently with the probability density function for the coordinates (x, y, φ) of each rectangle constant and equal to $\frac{1}{2}\, \pi\, R'$ in the three-dimensional interval with base R' and height π and zero outside this interval. In section 3 we evaluate the first two moments of the measure of X, where X denotes the intersection of the set-theoretical sum of the N rectangles ρ with R.

(ii) Let R denote the n-dimensional interval consisting of all points $(x_1, x_2, x_3, \cdots, x_n)$ such that $0 \leq x_i \leq A_i$, $(i = 1, 2, \cdots, n)$, and let R' denote the larger interval for which $-\delta \leq x_i \leq A_i + \delta$. Let a fixed number N of n-dimensional spheres with radii r (such that $2r \leq \min(A_i, 2\,\delta)$) be chosen independently, with the probability density function for the centre of each n-sphere constant and equal to $1/R'$ in R' and zero outside this interval. Denoting by X the intersection of the set theoretical sum of the N n-spheres with R, we evaluate in section 4 the first two moments of the measure of X. This problem is a generalization to n-dimensional space of the case considered by Robbins for the plane $(n = 2)$ in [4].

2. Preliminary formulas. Let K be an indeformable plane convex figure of variable position in the plane. The position of K may be determined by the coordinates (x, y) of a point P fixed within K and the angle φ which measures the rotation of K about P. We shall call x, y, φ the coordinates of K. The