NOTE ON DIFFERENTIATION UNDER THE EXPECTATION SIGN IN THE FUNDAMENTAL IDENTITY OF SEQUENTIAL ANALYSIS

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Let z be any chance variable and $z_1, z_2, z_3, \cdots$ a sequence of independent chance variables, each with the same distribution as z. Let $Z_N = z_1 + z_2 + \cdots + z_N$. Let $\phi(t) = Ee^{zt}$ for all complex t for which the latter exists. Let $S_1, S_2, \cdots$ be a sequence of mutually exclusive events such that S_j depends only on $z_1, z_2, \cdots, z_j$, and $\sum_{j=1}^{\infty} P(S_j) = 1$. Let the chance variable n be defined as $n = j$ when S_j occurs. Blackwell and Girshick [1], generalizing a result of Wald [2], showed that if there is a positive constant M such that

$$(1) \qquad |Z_N| < M \text{ when } n > N$$

then the identity

$$(2) \qquad E\{e^{Z_n t}(\phi(t))^{-n}\} = 1$$

holds for all complex t for which $\phi(t)$ exists and $|\phi(t)| \geq 1$. Wald [3] established conditions, including the existence of $\phi(t)$ for all real t, under which (2) may be differentiated under the expectation sign an unlimited number of times.

Without assuming the existence of $\phi(t)$ for a real t-interval the following result holds: *If (1) is true and if $E(z^k)$ and $E(n^k)$ are both finite, k a positive integer, then*

$$(3) \qquad E\left\{\frac{d^k}{ds^k}[e^{Z_n is}(\phi(is))^{-n}]_{s=0}\right\} = 0$$

where $i = \sqrt{-1}$ and s is real. Certain identities, obtained by differentiating (2) and putting $t = 0$, can also be obtained from (3). For example, if $En = 0$, and if En^2 and Ez^2 both exist then $EZ_n^2 = Ez^2 En$.

Let $P_N = P(n \leq N)$; $p_N = P(n = N)$. Let $H(j, Z_j)$ and $F(N, Z_N)$ be the conditional cumulatives of Z_j and Z_N for $n = j$ and $n > N$ respectively. Now (2) was derived by Wald [2], p. 285, from a relation, valid whenever $\phi(t)$ exists, which in the present notation becomes

$$(4) \qquad \sum_{j=1}^{N} p_j \int_{-\infty}^{\infty} (\phi(t))^{-j} e^{Z_j t}\, dH(j, Z_j) + \frac{(1 - P_N)}{(\phi(t))^N} \int_{-\infty}^{\infty} e^{Z_N t}\, dF(N, Z_N) = 1.$$

Examination of Wald's derivation of (4) shows it to be valid under the present hypotheses. Now the finiteness of $E(z^k)$ clearly implies that of $E(Z_j^k \mid n = j)$. Also, since $F(N, Z_N)$ is constant outside the interval $[-M, M]$, the integral $\int_{-\infty}^{\infty} Z_N^k\, dF(N, Z_N)$ is finite. Hence we may set $t = is$ in (4) and differentiate