NOTES

This section is devoted to brief research and expository articles on methodology and other short items.

A REMARK ON CHARACTERISTIC FUNCTIONS

By A. Zygmund

University of Pennsylvania

1. Let $F(x)$, $-\infty < x < +\infty$, be a distribution function, and

$$\varphi(t) = \int_{-\infty}^{+\infty} e^{itx} \, dF(x)$$

its characteristic function. It is well known that the existence of $\varphi'(0)$ does not imply the existence of the absolute moment

$$\int_{-\infty}^{+\infty} |x| \, dF(x). \tag{1}$$

A simple example is provided by the function

$$\varphi(t) = C \sum_{n=2}^{\infty} \frac{\cos nt}{n^2 \log n},$$

where C is a positive constant. Since the series on the right differentiated term by term converges uniformly (see [1]), $\varphi'(t)$ exists (and is continuous) for all values of t, and in particular at the point $t = 0$. Obviously $\varphi(t)$ is the characteristic function of the masses $C/2n^2 \log n$ concentrated at the points $\pm n$ for $n = 2, 3, \cdots$. The constant C is such that the sum of all the masses is 1. The divergence of the series $\Sigma 1/n \log n$ implies that in this particular case the moment (1) is infinite.

In a recent paper (see [2], esp. p. 120, footnote), Fortet raises the problem of whether the existence of $\varphi'(0)$ implies the existence of the first algebraic moment

$$\int_{-\infty}^{+\infty} x \, dF(x) = \lim_{X \to +\infty} \int_{-X}^{X} x \, dF(x). \tag{2}$$

The main purpose of this note is to show that this is so. We shall even prove a slightly more general result.

A function $\psi(t)$ defined in the neighborhood of a point t_0 is said to be *smooth* at this point if

$$\lim_{h \to +0} \frac{\psi(t_0 + h) + \psi(t_0 - h) - 2\psi(t_0)}{h} = 0.$$

Clearly, if ψ has a one-sided derivative at the point t_0, the derivative on the other side also exists and has the same value. Thus the graph of $\psi(t)$ has no angular point for $t = t_0$, and this explains the terminology. If $\psi'(t_0)$ exists and is finite, $\psi(t)$ is smooth for $t = t_0$. The converse is obviously false, since any

Institute of Mathematical Statistics is collaborating with JSTOR to digitize, preserve, and extend access to *The Annals of Mathematical Statistics.*
www.jstor.org