NOTES

This section is devoted to brief research and expository articles on methodology
and other short items.

A REMARK ON CHARACTERISTIC FUNCTIONS

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1. Let \( F(x), -\infty < x < +\infty \), be a distribution function, and
\[
\varphi(t) = \int_{-\infty}^{+\infty} e^{itx} \, dF(x)
\]
its characteristic function. It is well known that the existence of \( \varphi'(0) \) does
not imply the existence of the absolute moment
\[
\int_{-\infty}^{+\infty} |x| \, dF(x).
\]
A simple example is provided by the function
\[
\varphi(t) = C \sum_{n=1}^{\infty} \cos n t \quad n^2 \log n
\]
where \( C \) is a positive constant. Since the series on the right differentiated term
by term converges uniformly (see [1]), \( \varphi'(t) \) exists (and is continuous) for all
values of \( t \), and in particular at the point \( t = 0 \). Obviously \( \varphi(t) \) is the char-
acteristic function of the masses \( C/2n^2 \log n \) concentrated at the points \( \pm n \)
for \( n = 2, 3, \cdots \). The constant \( C \) is such that the sum of all the masses is 1.
The divergence of the series \( \Sigma 1/n \log n \) implies that in this particular case the
moment (1) is infinite.

In a recent paper (see [2], esp. p. 120, footnote), Fortet raises the problem of
whether the existence of \( \varphi'(0) \) implies the existence of the first algebraic moment
\[
\int_{-\infty}^{+\infty} x \, dF(x) = \lim_{x \to +\infty} \int_{-x}^{x} x \, dF(x).
\]
The main purpose of this note is to show that this is so. We shall even prove a
slightly more general result.

A function \( \psi(t) \) defined in the neighborhood of a point \( t_0 \) is said to be smooth
at this point if
\[
\lim_{h \to 0} \frac{\psi(t_0 + h) + \psi(t_0 - h) - 2\psi(t_0)}{h} = 0.
\]
Clearly, if \( \psi \) has a one-sided derivative at the point \( t_0 \), the derivative on the
other side also exists and has the same value. Thus the graph of \( \psi(t) \) has no
angular point for \( t = t_0 \), and this explains the terminology. If \( \psi'(t_0) \) exists and
is finite, \( \psi(t) \) is smooth for \( t = t_0 \). The converse is obviously false, since any

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